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CONDITIONS OF ELECTRIC EQUILIBRIUM AT BOUNDARY SURFACES; VOLTA EFFECT

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I shall here discuss more fully than I have done before the meaning and implication of two equations appearing in a paper which I published several years ago under the title *Thermoelectric Action with Dual Conduction of Electricity*.[1] These equations are

$$dP + dP_a = 0, \tag{1}$$

and

$$ne(dP + dP_f) = -dp, \tag{2}$$

and they were intended to express the conditions of electric equilibrium, no flow, at the junction of two metals, α and β, at the same temperature. The differential form is explained by the fact that, for the purpose of discussion, I imagined the two metals to be joined by an "alloy bridge" varying in composition from pure metal α at one end to pure metal β at the other end; the equations were supposed to hold for any point on this bridge.

The symbols in these equations require some explanation. P means electric potential in the ordinary sense, except as to the sign, which for all of the P's is so taken that an electron tends to move *down* the potential gradient. Thus e, the electron charge, is treated in my formulas as a positive quantity, so that any product eP will have the sign of the P factor.

P, according to my view of the matter, is due to what is called a surface charge on the metals, that is, an excess or deficiency of electrons for each metal as a whole compared with the number of positive ions for each metal as a whole. I am merely following unquestioned custom, based on Faraday's experiments, in assuming that any excess or deficiency of either kind of electricity shows as a surface charge, having no effect on the interior state of the metal except to raise or lower the potential everywhere. It seems to me plain that such a charge can not in itself be regarded as a *double layer*, and that, accordingly, there will be no sudden change of P

in passing from the surrounding air or vacuum into the metal, or *vice versa*. The question of a double layer at the junction of two different metals, due to their difference of surface charge, I shall take up later.

The subscripts a and f refer to the "associated" electrons and the "free" electrons, respectively, the associated electrons being, in my theory of conduction, electrons which in the case of electric flow pass from one atomic union to another—that is, from union with one positive ion to union with an adjacent positive ion—without getting into the "free" state. According to my investigations, or speculations, the greater part of the current in most metals is maintained by the associated electrons. The a electrons and the f electrons are subject alike to the potential, P, due to the surface charge as already defined.

On the other hand, the a electrons are subject to local attractions or repulsions, due to adjacent matter, which are independent of the surface charge that produces the potential P, and these local forces have a potential which I called, in the paper referred to, P_a. P_f has for the *free* electrons the same meaning that P_a, as just defined, has for the associated electrons.

Now P_a and P_f, being due to local attractions, are essentially different from P, as above defined, in this respect that each of them loses its value suddenly and completely when an electron passes from the interior of a metal to the air or vacuum outside. One may, when it seems worth while, think of this sudden change of potential as due to a double layer, the outward side of the surface atoms consisting of electrons, with the corresponding positive charges slightly inside the surface. Frenkel[2] has developed this idea, which seems to be valid and illuminating, but for my present purpose it need not be dwelt on. The point I wish to emphasize is this, that the surface jump of potential, the double-layer effect if we choose to consider it as such, has to do with P_a and P_f, not at all with P.

As to the so-called "image" or "mirror" effect upon an electron just outside a metal, it seems unnecessary for my present purpose to take this into account, as it depends merely on an induced charge and is therefore the same for one metal as for another.

If we now, using the subscripts α and β to indicate the two metals dealt with, integrate equation (1) from the pure α end to the pure β end of our "alloy bridge," we get

$$(P + P_a)_\alpha = (P + P_a)_\beta. \qquad (3)$$

I must now give the reasoning on which equation (1), and so equation (3), is based.

Let us suppose that $(P_a)_\alpha$ is less than $(P_a)_\beta$, α and β being, let us say, copper and zinc, respectively, so that work is required to carry an electron from atomic union in copper to atomic union in zinc; and let us suppose the two metals joined, but with, at first, no surface charge, no excess or

deficiency of electrons, in or on either metal, so that P is for the moment zero for each. In any encounter between a copper atom and a zinc ion, at surface of junction (the "alloy bridge," a temporary structure for theory building, is now abolished), the copper atom will be likely to retain its full complement of electrons, so that the zinc ion will remain an ion. But when a zinc atom has contact with a copper ion, there is a chance at least that the superior attraction of the copper ion will prevail, so that it will gain an electron and become an atom, leaving its zinc antagonist in the ion state. Thus a difference of *charge*, and so a difference of P, for the two metals begins, and this process will continue until a state of equilibrium is established between the a electrons of the two metals.

Now what is the condition for this state of equilibrium? In writing equation (1) or equation (3), I have assumed that equilibrium will be reached when the total potential energy, due to the surface charge potential P and the local attraction potential P_a, become the same for an associated electron of α as for an associated electron of β.

This seems plausible at first, but in making this assumption I have ignored the fact that the associated electrons may have an appreciable amount of kinetic energy, and that this amount may not be the same in the case of metal α as in the case of metal β. At the suggestion of my colleague Doctor Kemble, I have now decided to take explicit account of this kinetic energy, which will be called k with the appropriate subscript. Accordingly $e(P_a) + k$ will indicate the total energy, potential and kinetic, which an associated electron has in virtue of being such—that is, all the energy the associated electron has in addition to the amount eP due to the surface charge on the metal.

Of course the potential energy eP_a is always negative, and the more strongly the electron is bound to the other parts of the atom the greater, numerically, is this negative potential energy. But evidently the kinetic energy of the associated electron has a tendency to release the electron—that is, the positive kinetic energy neutralizes in part the negative potential energy. Accordingly the simplest assumption we can now make as to the condition for equilibrium is that the *total energy* of an associated electron of α shall be the same as the total energy of an associated electron of β. This gives us the equation

$$e\,(P + P_a)_\alpha + k_\alpha = e\,(P + P_a)_\beta + k_\beta. \qquad (3')$$

Let us now suppose that we have some means of taking an electron out of an atom in metal α and carrying it completely clear of this metal into any standard condition where its energy will be F. The gain of energy of the electron in this operation, and so the amount of energy needed to accomplish it by a reversible process, is $F - (e\,(P + P_a)_\alpha + k_\alpha)$. The corresponding quantity for an associated electron of β is $F - (e\,(P + P_a)_\beta$

$+ k_\beta$). But if equation (3′) holds these two quantities of energy are equal. That is, when two metals α and β are in metallic connection with each other, the amount of energy needed to expel an associated electron from α is the same as the amount of energy required to expel an associated electron from β, if in each case the expelled electron reaches the same final condition, say lodgment in a third metal, C.

It seems to be generally admitted that the photo-electric emission current consists of electrons which have been taken not from the "free" state within the emitting metal but from the atoms themselves, the absorbed radiation energy $h\nu$ doing the work of detaching an electron from the periphery of an atom and taking it out of the metal with a greater or less kinetic energy. A method of experimentation is indicated diagrammatically in figure 1, the exciting radiation being represented, for convenience, as entering the rear of the metal plate instead of the emitting surface. Here α and β are two metals in contact with each other and with a third metal, C.

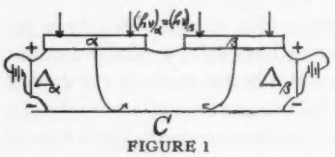

FIGURE 1

Both α and β are subjected to radiation of a definite frequency, and this frequency is the same for each, so that $(h\nu)_\alpha = (h\nu)_\beta$. Under this excitation each metal sends photo-electrons toward C. The potential difference Δ_α, introduced between α and C according to the indications of the diagram so as to make α positive with respect to C, is just sufficient to prevent the photo-electrons of α from reaching C; and Δ_α is called the "stopping potential" for α. In like manner Δ_β is the stopping potential for β.

Millikan,[3] using substantially this method, confirms the indications previously found by Richardson and Compton[4] and by Hennings[5] and declares that the stopping potentials are the same for different metals—that is, $\Delta_\alpha = \Delta_\beta$, whatever the metals α and β may be.

Professor Millikan, in the paper referred to, appears to be somewhat at a loss to explain this equality, but it is precisely what equation (3′) would lead us to expect. We have in this experiment, according to the preceding argument,

$$e(P + P_a)_\alpha + k_\alpha + (h\nu)_\alpha - \Delta_\alpha = e(P + P_a)_\beta + k_\beta + (h\nu)_\beta - \Delta_\beta, \quad (4)$$

and, as $(h\nu)_\alpha = (h\nu)_\beta$, equation (3′) taken with (4) requires that $\Delta_\alpha = \Delta_\beta$.

If an electron photo-electrically expelled from α were stopped just far enough outside the surface of the metal to be free from the local attractions and repulsions that determine $(P_a)_\alpha$, it would still be subject to the attraction or repulsion of the *surface* charge of α and would accordingly still have the potential energy eP_α due to this charge. A corresponding electron from β would similarly have still the potential energy eP_β. The

two potentials P_α and P_β are different, and their difference is the observed "Volta effect."

Of course, in giving this explanation of the Volta effect, as due to a difference of surface charge caused by the contest between α ions and β ions for possession of electrons at the junction of the two metals, I am merely giving to an old idea, the idea of specific attractions of metals for electricity, a new and definite form, by bringing into action the positive ions, so long and so strangely neglected by other writers.

But it may well be asked whether I have not in the preceding discussion unwarrantably ignored the free electrons within the metal, and I must now deal with them. As I have indicated above, they, as constituting a gas, are subject to a gas pressure, p, which may tend to drive them from one place to another, so that their movements are not to be inferred from the potential gradients only to which they are subject. They, in fact at any boundary surface, whether it be between the two metals α and β or between either metal and the almost vacuous space outside it, obey the equilibrium law of Boltzmann, difference of electron gas-pressure on the two sides of the boundary off-setting difference of potential.

Putting nRT for p in equation (2) and integrating from the α end to the β end of the alloy bridge, I get the Boltzmann equation

$$\frac{n_\alpha}{n_\beta} = exp\left[-e\frac{(P+P_f)_\alpha - (P+P_f)_\beta}{RT}\right] \tag{5}$$

Indicating by n'_α and n'_β the number density of the free electrons just outside α and β, respectively, we have for α

$$\frac{n'_\alpha}{n_\alpha} = exp\left[-e\frac{P_\alpha - (P+P_f)_\alpha}{RT}\right] = exp\left[e\frac{(P_f)_\alpha}{RT}\right], \tag{6}$$

and for β

$$\frac{n'_\beta}{n_\beta} = exp\left[e\frac{(P_f)_\beta}{RT}\right]. \tag{7}$$

Dividing (6) by (7) we get

$$\frac{n'_\alpha}{n'_\beta} \times \frac{n_\beta}{n_\alpha} = exp\left[e\frac{(P_f)_\alpha - (P_f)_\beta}{RT}\right]. \tag{8}$$

Multiplying (8) by (5) we get

$$\frac{n'_\alpha}{n'_\beta} = exp\left[-e\frac{P_\alpha - P_\beta}{RT}\right]. \tag{9}$$

Equation (9), which refers to conditions just outside the two metals, is obviously the proper expression for equilibrium of the atmosphere of electrons extending, with varying density, from one metal to the other.

For certain purposes it may be necessary to multiply n_α and n_β each by some factor greater than unity, in order to take account of the fact that the n free electrons per unit volume of a metal do not have the whole of this volume to move about in; but this refinement, impossible of definite application as yet, has not seemed advisable in the present discussion.

I must now return to the question raised earlier in this paper as to whether, if the surface charge on which P depends does not constitute a "double layer" at the surface between a metal and a vacuum, there can be a double layer due to surface charge at the junction surface of two metals. In figure 2 there is a charge on each of the opposed surfaces of the plates α and β. We do not call these two opposite charges a double layer, in the technical sense, so long as the distance between them is considerable. Let the two opposing surfaces touch, however, and the two charges, grown large enough to maintain the same difference of potential $(P_\alpha - P_\beta)$, as before, do constitute a double layer in the accepted sense of the term.

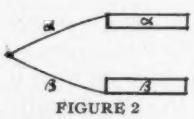

FIGURE 2

If I am asked to state the condition of equilibrium which exists between the a electrons within a metal and the free electrons of the "atmosphere" just outside the metal, I can only say that, as the f electrons within a metal are in equilibrium with the a electrons there according to the mass law and with the free electrons outside according to the Boltzmann equation, equilibrium between the a electrons and the free electrons outside appears to be assured. An attempt to revise the Boltzmann equation in order to take account of the possible occasional spontaneous passage of an electron from the a state directly to the free state outside the metal, or vice versa, would seem to me like trying to revise the Boltzmann equation for atmospheric equilibrium in such a way as to take account of the possible influence of gas particles adsorbed or absorbed by solids in the atmosphere.

[1] *Proc. Nat. Acad. Sci., Washington,* **4** (1918), pp. 98–103.
[2] *Phil. Mag., London,* **33** (1917), pp. 297–322.
[3] *Physic. Rev., Ithaca,* **18** (1921), p. 236.
[4] *Phil. Mag.,* **24** (1912), p. 575.
[5] *Physic. Rev.,* **4** (1914), p. 228.

THE EFFECT OF A SURROUNDING BOX ON THE SPECTRUM OF SCATTERED X-RAYS

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In the September number of these PROCEEDINGS Professor Duane and his collaborators have described experiments in which differences occurred in the ionization spectra obtained from sulphur traversed by X-rays from an X-ray tube with a molybdenum target depending upon the presence of a wooden box surrounding the X-ray tube and the sulphur radiator.[1] The fact that the spectra obtained in the presence of the box were similar to those obtained in this laboratory, whereas those obtained in the absence of the box did not show what we call the "modified lines" suggested to these experimenters that the presence of these lines was dependent upon the presence of the box. If this is true, the origin of these lines must be different from that postulated in the quantum theory of X-ray scattering. It was suggested rather that the modified lines were due to a "tertiary radiation" from the carbon in the wood box.

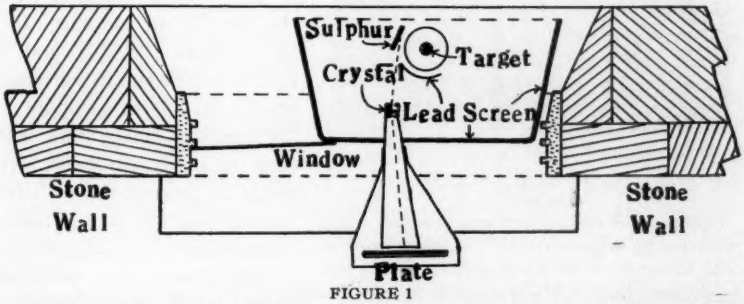

FIGURE 1

The X-ray tube and the sulphur scattering block were placed outside of a window, with a lead screen partially surrounding the X-ray tube to prevent any X-rays from entering the window.

Before our experiment was performed, Y. H. Woo compared the spectra obtained by the ionization method when a lead-lined box surrounded his X-ray tube and radiator with those he previously obtained on using a wood-lined box.[2] The comparison was made for magnesium and aluminium radiators, and new measurements were made on silicon and sulphur radiators. In every case both the modified and unmodified lines appeared with substantially the same intensity as when the wood-lined box was used, and in approximately the positions demanded by the quantum theory.

In order to satisfy ourselves completely, we set up a water cooled molybdenum target X-ray tube and a sulphur radiator outside a third floor window, with no surrounding box, as shown diagrammatically in figure 1. The face of the target pointed upward, so that nearly all of the rays which did not strike the lead screen shielding the window went to the open sky. The spectrum was obtained photographically, using a modified Seeman spectograph. The tube was operated continuously at about 25 m. a. and 47 k. v. peak.

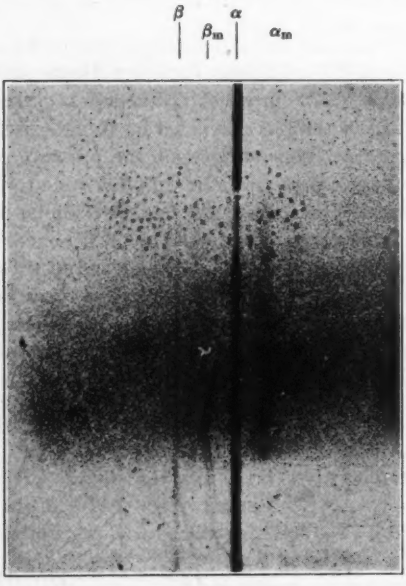

FIGURE 2

Spectrum of X-rays from molybdenum target scattered by sulphur at 118° (middle) compared with spectrum of the fluorescent rays from molybdenum (edges). The presence of the modified as well as the unmodified lines is evident for the scattered rays.

To obtain suitable reference lines on our film, and to be sure that no false lines appeared due to faults in the crystal, the edges of the film were first exposed for 8 hours to the fluorescent rays from a sheet of molybdenum slipped over the face of the sulphur radiator. Thus the $K\alpha$ and β lines of molybdenum were registered, as shown in figure 2. The middle of the film was then exposed for about 170 hours to the secondary rays coming from the sulphur itself. In addition to the unmodified α and β lines, this spectrum shows plainly the modified α line and very faintly the modified β line.

The angle of scattering of the α line was 118 ± 3 degrees, whence according to the quantum formula the wave-length change should be $\delta\lambda = 0.036 \pm 0.001\ A$. Measurements on the $K\alpha$ lines showed $\delta\lambda = 0.037\ A$, which is a satisfactory agreement.

The relative energy in the modified and unmodified lines of this photographic spectrum did not differ much from that shown by Woo's ionization spectra taken with a surrounding box. We are thus unable to find any considerable effect on the character of the spectrum obtained due to the presence of a surrounding box.

[1] Armstrong, A. H., Duane, W., and Stifler, W. W., *Proc. Nat. Acad. Sci.*, **10**, 374 (1924).

Allison, S. K., Clark, G. L., and Duane, W., *Ibid.*, p. 379 (1924).

[2] Woo, Y. H., this number of these PROCEEDINGS.

THE SIGNIFICANCE OF THE DISCOVERY OF X-RAY LAWS IN THE FIELD OF OPTICS

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Communicated December 19, 1924

I. *The Extension of Moseley's Law to the Field of Optics.*—Through our recent stripping of all the valence electrons from one up through seven from the whole group of atoms, sodium, magnesium, aluminum, silicon phosphorus, sulphur and chlorine, and from one up through five from the group, lithium, beryllium, boron, carbon and nitrogen, we have obtained for the first time a long series of light atoms having an identical electronic structure, but a linearly increasing nuclear charge. It is precisely this combination of identity of internal electronic structure among heavy atoms with linearly increasing nuclear charge which is responsible for the existence of the Moseley Law in the X-ray field, and the so-called irregular-doublet law, which flows from it.

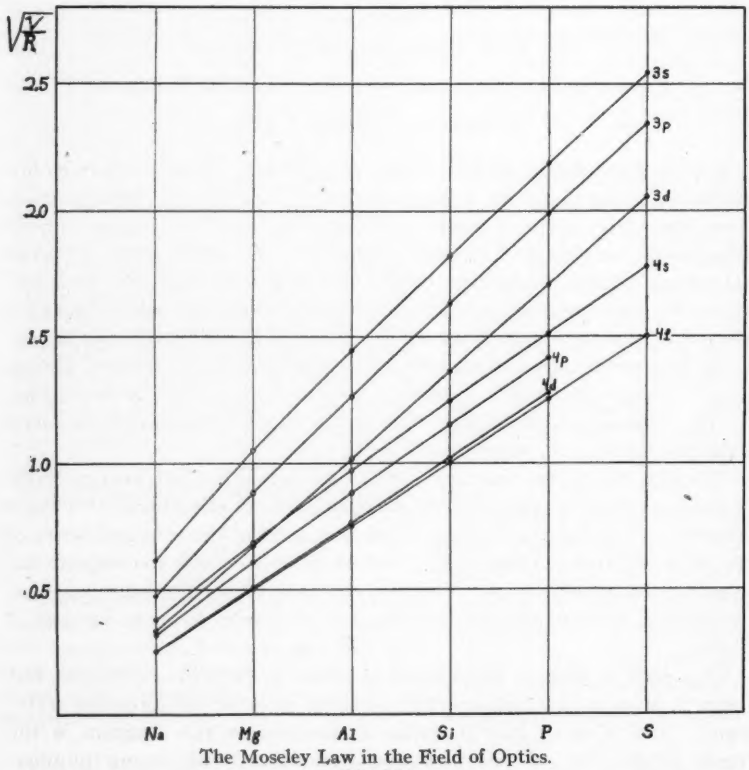
Through the recent working out[1] of all the important energy levels (term-values) of the spectra of the stripped atoms of phosphorus (*P V*) and sulphur (*S VI*) and the locating of the first term of the principal series of the stripped atom of chlorine (*Cl VII*) we have been able to complete the experimental proof *that the Moseley Law, with its corollary the irregular-doublet law, holds in the field of optics just as beautifully as in the field of X-rays.*

This work is soon to be reported in detail in the *Physical Review*, but figure 1, gives a very illuminating graphical proof of the foregoing statement. The Moseley Law is shown everywhere in this diagram in the linear progression of our measured term-values with atomic number. The irregular-doublet law is seen in the parallelism between the lines joining the 3*s* terms, the 3*p* terms and the 3*d* terms, or the 4*s*, the 4*p*, the 4*d* and the 4*f* terms.

When it is remembered that these terms are all in the visible or the ultra-violet region explored by ordinary grating methods, it will be seen that the proof is complete that the Moseley Law is quite as much an optical as an X-ray law.

II. *A Possible Interpretation of the Relativity Doublet Law.*—Our fur-

ther recent discovery,[2] that the regular or relativity doublet law also holds throughout the field of optics, instead of fitting in beautifully with theory as does the extension to the field of optics of the Moseley and Hertz laws, is extremely difficult to reconcile with the present status of physical theory. Indeed, we have recently shown[3] that the exact validity of this law in optics, if granted, forces us to choose between one or the other of two horns of a dilemma which may be stated thus;



The Moseley Law in the Field of Optics.
FIGURE 1

(1) The abandonment of relativity causes and effects altogether in electronic orbits or

(2) The abandonment of Bohr's interpenetration ideas and with them the practice of assigning azimuthal quantum numbers 1, 2, 3, 4, etc., to *s*, *p*, *d*, *f* terms, respectively.

For reasons which will be discussed more fully in a forthcoming article,[4] *both of these horns seem to us completely inadmissible.*

The only possible escape then was to discredit the completeness of our proof of the *exact* applicability of the relativity doublet law to the explanation of fine structure generally. This is what we have recently endeavored to do, both by further analysis and by further experimental work. The results of this study may be briefly summarized thus:

(1) The relativity doublet law does indeed account for fine structure in optics quite as well as in the field of X-rays. Combined with the Moseley law, it also has provided us with one of the most powerful tools thus far discovered for identifying the origins of spectral lines and working out spectral series.

(2) In neither the field of optics nor the field of X-rays, save in the case of atomic hydrogen and ionized helium (H and He^+) does the theoretical relativity doublet law, numerical constants and all, predict the observed fine-structure-separation within quite the limits of observational error. There must always be introduced a so-called screening constant s, which is essentially an arbitrary parameter with the aid of which the experimental and theoretical results are made to agree.

(3) It is a very remarkable fact, however, that in the three or four limiting cases in which we know what this screening constant ought to be, the relativity law applied to measured doublet separations yields *nearly* the correct value of s, but not quite so within the limits of observational error.

(4) This last fact furnishes some little justification for finding a break in the continuity of our proof of the quantitative applicability of the relativity formula to the fine structure of the spectra of all atoms from hydrogen to uranium. This break we make between ionized helium and lithium and hence consider the cause of the fine structure of atomic hydrogen and ionized helium to be the relativity cause as has been customary heretofore; for this cause predicts quantitatively within the very narrow limits of observational error the observed separations.

(5) All other so-called relativity doublets, both the familiar ones in X-rays and our own newly discovered ones in the field of optics, we assume to be, in fact, not relativity doublets at all, despite their general following of the relativity law. We assign them instead to some new cause, of magnetic origin, or a combination of magnetic and electrostatic origin, which operates between the nucleus and two orbits of the same shape but different orientations with respect to the nucleus, and which not only follows very closely the same law of variation with atomic number (a fourth power law), with total quantum number (a third power law) and with azimuthal quantum number as does the relativity cause, but which has also nearly the same numerical constants as has that law.

The chief advantage in this apparently strange and difficult assumption of a fairly close accidental agreement between the effect of this new hy-

pothetical cause and the relativity cause is that, if we have no alternative between throwing overboard relativity as a cause altogether and postulating some other cause which, quite accidentally, leads to a formula in *exact quantitative* agreement with the relativity formula—such a postulate is necessary if the behavior of atomic hydrogen and ionized helium are to be placed within the scope of the new cause—or, on the other hand, postulating a new cause upon which the demands are a little less rigidly quantitative, even though still quite severe, we must of course make the second and less extreme choice; *for precise quantitative agreement between theory and observation constitutes by far the best evidence which we can have at all for the correctness of any of our physical conceptions.*

A crucial test of this assumed fundamental difference between the nature of the fine structure of the lines of hydrogen and ionized helium and those of all other atoms, ought to be found by comparing the behavior of these two different types of substances in magnetic fields. *In weak fields this sharp difference is actually manifested.* For hydrogen and ionized helium show the normal Zeeman effect, while lithium and the like show the anomalous Zeeman effect. In strong magnetic fields, on the other hand, the Paschen-Back effect, exhibited by lithium and the like as it should be, should not be exhibited by hydrogen and helium at all. This last crucial point has not yet been decided to a certainty, we think, by the experimentalist, though the evidence up to date is in favor of the existence of the Paschen-Back effect in the spectra of hydrogen and helium.[5]

Nevertheless, the evidence for a difference in kind between the fine structure of the lines of the two lightest elements, atomic hydrogen and ionized helium and that of lithium and all the elements beyond it is so good, despite their apparent similarity in behavior in strong magnetic fields, that we propose to attribute the doublets of atomic hydrogen and ionized helium to a true relativity cause, and to introduce a new non-relativistic cause, which, however, obeys an equation almost exactly like the relativity equation, to account for the behavior of lithium and the elements of higher atomic number. This is the only possible way to retain both Bohr's interpenetration ideas and Sommerfeld's relativistic treatment of electron orbits, and both of them seem at present to demand retention. To find a new cause for the relativity-doublet formula with only a little leeway in the value of the numerical constants is a problem worthy of the efforts of the theoretical physicist.

[1] Bowen and Millikan, *Physic Rev.* (in Press).

[2] Bowen and Millikan, *Ibid.*, Sept. 24, 1924, p. 209.

[3] Millikan and Bowen, *Ibid.*, Sept. 24, 1924, p. 228.

[4] Millikan and Bowen, "A Possible Reconciliation of Bohr's Interpretation Ideas with Sommerfeld's Relativistic Treatment of Electron Orbits."

[5] Oldenburg, *Ann. Physik* **67**, 253 (1922); Försterling and Hansen, *Zeit. Physik*, **18**, 26 (1923).

THE COMPTON EFFECT AND TERTIARY X-RADIATION

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Communicated January 7, 1925

This research is a continuation of that by Compton and Woo[1] and was performed to obtain data which bear upon the important question under discussion at the present time, namely, the nature of the change of wave-length due to the scattering.

The exact arrangement of the apparatus was described in the note referred to.[1] The water-cooled molybdenum target tube was operated at about 60 kilovolts peak and 50 milliamperes. In the present work, however, instead of a wood box covered with lead to contain the X-ray tube, the box was completely lined with $^1/_{16}$ inch lead sheet. This was designed to avoid a possible box effect, as described by Allison, Clark and Duane,[2] due to the secondary radiation coming from the carbon and oxygen atoms composing the wooden walls of the box.

The secondary radiators used were rock-salt, magnesium, aluminium, silicon and sulfur. They were all in the form of flat plates. The rays scattered from rock-salt may be regarded characteristic of sodium and chlorine.

The results of the experiments are shown in figure 1. The spectra from sodium, magnesium and aluminium are identical in character with those obtained by Compton and Woo[1] with the X-ray tube in a wood box. The box effect is thus not detected by the present work. The spectra from silicon and sulfur show in each case an unmodified line P occurring at the same position as the fluorescent Mo $K\alpha$ line and a modified line whose peak M is within experimental error at the position predicted by Compton's theory.[3] A detailed examination of the box effect and the confirmation of the wave-length shift in the case of scattering from sulfur by a photographic method is described in the following note by Prof. Compton and Mr. Bearden.

Recently Clark, Duane and Stifler[4] have published accounts of experiments on measurements of the wave-lengths of molybdenum $K\alpha$ rays scattered from ice, rock-salt, aluminium and sulfur. Their results indicate the presence of tertiary radiation whose minimum wave-length is $\lambda\lambda_k/(\lambda_k - \lambda)$, where λ is the wave-length of the incident rays and λ_k is the critical K absorption wave-length of the scattering element.

In figure 1, T marks on each curve the position of the short wave-length limit of the tertiary radiation for the corresponding scattering element. Under the condition of the present experiment the writer found, in every case of the five radiators used, no evidence for the peak of the

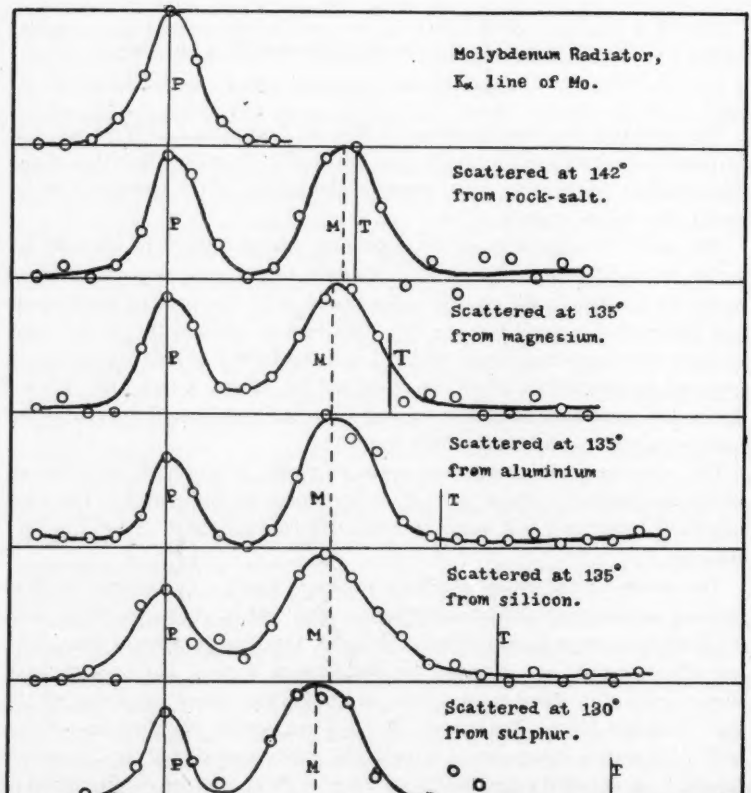

FIGURE 1

Spectra reflected from calcite crystal of the secondary radiation from various elements, traversed by X-rays from a Mo target. *P* marks the position of the primary K_α line, *M* the position of the modified peak calculated from Compton's theory, and *T* the position of the short wave-length limit according to the tertiary ray idea.

tertiary ray as observed by Clark, Duane and Stifler in their experiments.[4] Since the presence of the tertiary ray from sulfur appears to be confirmed by experiments reported more recently,[2] the writer in particular performed ten experiments with sulfur as the secondary radiator (the curve for sulfur in figure 1 represents one of the results). None of these experiments showed the existence of this tertiary peak.

The writer is indebted to Prof. A. H. Compton for his interest in this work.

[1] Compton and Woo, these PROCEEDINGS, **10**, 370 (1924).
[2] Allison, Clark and Duane, these PROCEEDINGS, **10**, 370 (1914).
[3] A. H. Compton, *Physic. Rev.*, **21**, 483 (1923).
[4] Clark, Duane and Stifler, these PROCEEDINGS, **10**, 148 (1924).

THE AGE OF THE STARS

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Read before the Academy November 11, 1924

It is well-known that the supply of energy made available by the gravitational contraction of the material of a star is insufficient, in view of the Stefan law of total radiation, to allow the time-life of a star to be more than a small fraction of the age demanded by geological considerations. Discussions of the problem of maintenance of stellar energy have been given by Shapley[1] and Russell.[2] The former discusses an hypothesis of asymmetrical radiation flow in which the rate of radiation to empty space is much less than the radiation toward other matter. Shapley also considers the destruction of mass as a possible source of stellar energy. He concludes that "... it now appears that the disagreement between the long and short time scales must be decided in favor of an exceedingly prolonged history for sidereal systems, permitting a relatively slow evolutionary development for stars and planets." Russell's paper is a very general sketch of the main outlines of the problem and is concerned mainly with the conditions under which a nuclear "unknown" source of energy comes into action.

In this paper, it is shown that the relativistic relation between energy and mass leads directly to a means of estimating the age of the stars, the method being independent of the atomic processes whereby mass as "matter" is changed into mass as "radiant energy." This relation,

$$\Delta E = c^2 \Delta m \qquad (1)$$

in which E is the energy of the system in ergs, c the fundamental constant

of space-time in centimeters per second, and m is mass in grams, has been given an interpretation that is too narrow, at least in discussions of its possible applications to stellar problems. It is necessary to bear in mind that, from the manner of its derivation, the equation necessitates that loss of mass by a system accompany loss of energy as a result of radiation. We do not need to inquire into the source of the energy lost by radiation.

The point is brought out clearly in the original work of Einstein.[3] He says, "Gibt ein Körper die Energie L in Form der Strahlung ab, so verkleinert sich seine Masse um L/V^2." And again, "Die Masse eines Körpers ist ein Mass für dessen Energieinhalt; ändert sich die Energie um L, so ändert sich die Masse in demselben Sinne um $L/(9 \times 10^{20})$ wenn die Energie in Erg und die Masse in Grammen gemessen wird."

Let us assume that the amount of mass which is gained by a star during its life by means of the actual accumulation of meteoric matter is negligible and further, that there is no expulsion to infinity of matter by the star during its life. If a star is born by the accumulation of dispersed matter, the first of these assumptions is probably inadmissible in the early stages of its life, but will be true with increasing accuracy as the star gets well along in years. The second assumption is probably true at all times as the explosive disruption of a star is considered to be an exceptional occurrence.

If we grant these assumptions, at least after the period of infancy, it is evident that since there is a continual loss of energy by radiation, the mass of the star will decrease throughout its life.

The rate of radiation of energy may be taken to be in accordance with Stefan's law. This allows the equation,

$$dE/dt = 4\pi r^2 \sigma T^4 \tag{2}$$

in which dE/dt is the rate of loss of energy, r the radius of the star, σ the Stefan constant, T the effective temperature, all in C.G.S. units. Equation (1) in the derivative form is

$$dE/dt = c^2(dm/dt). \tag{3}$$

Combining (2) and (3) we may write,

$$dt = \frac{c^2 dm}{4\pi\sigma r^2 T^4} \tag{4}$$

which is the fundamental equation of this paper. It expresses the lapse of time corresponding to a differential decrease of mass in terms of known quantities.

For the computation it was convenient to use other units. Let τ be time in years, M be mass with the Sun's mass as unit, and R be the star's radius with the Sun's radius as unit. Then we have

$$\Delta t = 3.15 \times 10^7 \ \Delta\tau, \qquad m = 1.99 \times 10^{33} \ M, \qquad r = 6.95 \times 10^{10} \ R.$$

Also we have

$$\sigma = 5.72 \times 10^{-5}, \qquad c = 3.00 \times 10^{10}.$$

Using these figures, (4) gives the relation,

$$d\tau = 1.63 \times 10^{28} dM/R^2T^4 \tag{5}$$

The division of stars of given spectral type into two classes according to absolute magnitude (giants and dwarfs) presented by Russell[4] is the basis of modern theories of stellar evolution. It is usually inferred that the course of evolution of individuals is along the lines of maximum frequency on the diagram of spectral type against absolute magnitude; the individual makes his debut as a giant of M-type and roughly zero absolute magnitude proceeds to B-type at approximately constant absolute magnitude, then declines from B to M while losing in luminosity so that at M-type its absolute magnitude is about 9 or 10. But there is considerable question whether each individual actually follows this course.

Eddington[5] has shown on theoretical grounds that the maximum temperature attained by a star will depend on the mass of the star and that only the more massive stars will attain B-type. Those of lesser mass will start to decline after having reached only A or F type.

If for some reason the stars all start with approximately the same mass and this mass is sufficient to carry them to B-type, the path of individuals on the Russell diagram will be simply that of the maximum frequency on the diagram. But if there is dispersion in the initial values of the mass then some individuals will short-circuit the main path, appearing on the dwarf branch without having attained B-type. Such a short-circuiting accounts for the presence of a considerable number of individuals between the giant-line and the dwarf-line of the diagram.

In the very comprehensive discussion of the present state of knowledge of masses of the stars by Seares,[6] it is brought out that the observational data on the masses of the giants are meager. The mean masses of stars on the dwarf-line, however, seem well-defined. We may ignore, in the absence of an exact method of allowing for it, the effect of short-circuiting by small stars. This short-circuiting will affect the data in such a way as to make the decrease of mass of an individual in going from B to M type appear greater than it really is and so to make the time scale too long. Ignoring this effect, Seares' data allow the evaluation of a rough time scale along the dwarf branch.

From table XII of Seares' paper were taken the values for the effective temperature of the dwarfs of each spectral type, assuming this to be the same as for the giants in the range B_0 *to* F_0 inclusive. From Seares' table XIV were obtained the values of R $(=D)$ for each type and from his table XXIII were obtained the mean values for the mass of each type.

From these data was computed the value of the integrand in (6) for each spectral type. From the values so obtained was formed a column giving the arithmetic mean of the values of the integrand for each adjacent pair of types. These means were multiplied by the corresponding ΔM to effect an approximate integration by the trapezoidal rule. Taking the origin of time as the instant when the star is of type G_0, to correspond with the choice of the Sun as unit, the integrated time lapses were computed by summing the separate time intervals.

The results of the computation follow:

TYPE	v	M	ΔM	$\bar{y}\Delta M$	$\Sigma\bar{y}\Delta M$	TIME (YRS.)
B_0	1.78	10.	1.7	7.4	−564.	-9.2×10^{12}
B_5	6.92	8.3	2.3	28.1	557.	9.06
A_0	17.5	6.0	2.0	57.8	529.	8.62
A_5	40.1	4.0	1.5	108.	471.	7.67
F_0	104.	2.5	1.0	165.	363.	5.92
F_5	226.	1.5	0.5	198.	−198.	−3.23
G_0	568.	1.0	0.24	210	0.	0.
G_5	1180.	0.76	0.08	136.	+210.	+3.42
K_0	2220.	0.68	0.06	217.	346.	5.64
K_5	5030.	0.62	0.03	492.	563.	9.18
Ma	27900.	0.59			+1055.	+25.2

In the table, y is equal to $10^{18}/R^2T^4$. The time in years in the last column is obtained by multiplying the quantities $\Sigma\bar{y}\Delta M$ by 10^{-18} and also by 1.63×10^{28}, the factor of equation (5).

In figure 1, M is plotted as a function of the time on the basis of the third and seventh columns of the table.

There is a correspondence between the results here and some given by Shapley and Miss Cannon[7] that is worth mentioning even though it leads to considerations which cannot be developed further at present. One might suppose, at first sight, that the number of stars in a given volume of space of each spectral type would be proportional to the amount of time that each individual spends in the several spectral classes. This is the case, at least roughly, for Shapley and Miss Cannon,[7] from a study of the recently-completed Draper catalog, find the following figures for the number of dwarfs of the several types in a million cubic parsecs of the space around the solar system:

$$B, 4.4;\ A, 250;\ F, 680;\ \text{and}\ G, 7600.$$

Although further figures are not given, they remark, "Dwarf stars of classes K and M are probably much more numerous per unit volume than dwarf stars of class G... The most important deduction from the table above is that the great majority of the stars are extreme dwarfs apparently indicating that a star spends most of its life in the later stages of the spectral series."

It is really doubtful whether this important deduction can be made from these data alone. The relative frequencies in the sky today depend on three things: 1, the relative duration within each spectral class; 2, the distribution in space of the birthplaces of the stars and in time of the birthdays of the stars; and 3, the motions of the stars with respect to the solar system in the course of their evolution. The exact analysis of the effect of these three causes does not seem to have been given. If the chaos of the motions makes the third factor ineffective and if we can assume a constant "birth-rate" among the stars, then (and only then, I think) is the deduction of Shapley and Miss Cannon allowable.

The relative duration of the stars within the various spectral types must be related in some way to the relative frequency of stars of each spectral

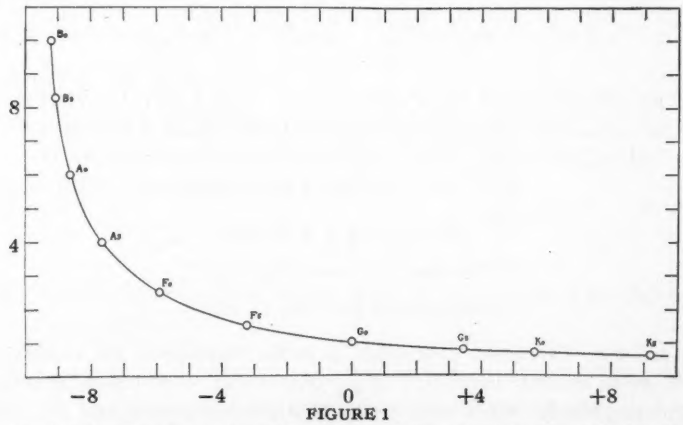

FIGURE 1

Mass of dwarf stars as a function of the time, the unit of the time (abscissæ) is 10^{12} years; that of the mass (ordinates) is the Sun's mass.

type as found in the sky today. If we suppose that new stars are being born at a steady rate and old stars are dying at a steady rate so that the population of the stellar universe is approximately constant, then we should expect to find the actual number of stars of each spectral class in a given volume at any time to be exactly proportional to the relative duration of an individual within each class.

Comparison with observation, however, will neither support nor discredit any particular calculation of relative duration in each spectral class. From exact knowledge of relative duration and from exact observation of the relative frequency in the sky today, we can only infer the distribution in time of the stellar birthdays.

While, therefore, no great weight is to be attached to the comparison, it

is of interest to note that the frequency of the dwarfs of various types in a million cubic parsecs around the solar system, corresponds roughly to the relative durations as computed here.

[1] Shapley, Harlow, *On Radiation and the Age of the Stars. Pub. Astron. Soc. Pac.*, **31**, 178 (1919).

[2] Russell, H. N., *On the Sources of Stellar Energy. Ibid.*, **31**, 205 (1919).

[3] Einstein, A., *Ist die Trägheit eines Körpers von seinem Energienhalt abhängig? Ann. Physik*, **18**, 639 (1905).

[4] Russell, H. N., *Relations between Spectra and Other Characteristics of the Stars. Pub. Amer. Astron. Soc.*, **3**, 22.

[5] Eddington, A. S. *On the Radiative Equilibrium of the Stars. Zeit. Physik*, **7**, 351 (1921).

[6] Seares, F. H. *The Masses and Densities of the Stars. Astrophys. J.*, **55**, 165 (1922).

[7] Shapley, H., and Cannon, A. J. *Summary of a Study of Stellar Distribution. Proc. Amer. Acad. Arts Sci.*, **59**, no. 9 (1924).

NOTES ON STELLAR STATISTICS. III: ON THE CALCULATION OF A MEAN ABSOLUTE MAGNITUDE FROM APPARENT MAGNITUDES, ANGULAR PROPER MOTIONS AND LINEAR RADIAL VELOCITIES

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A problem of frequent occurrence in stellar statistics is the calculation of the mean absolute magnitude of a group of stars of widely different apparent magnitudes when only the angular proper motions and the linear radial velocities are known. In what follows we shall assume that a solution for solar motion has been made and that the proper motions have been resolved into the v components parallel to the sun's motion, and the τ components at right angles thereto and that the radial velocities have been corrected for the influence of this solar motion.

A procedure generally followed is the reduction of all proper motions to one and the same apparent magnitude, m_o. The arithmetic mean value of the reduced τ components is then compared with that of the radial velocities. Likewise, the algebraic mean of the v components is compared with the total speed of the sun. Both comparisons will yield a value for the mean parallax of these stars, but, owing to the dispersion in absolute magnitude and in linear velocity the method is not entirely flawless. However, when sufficient material is at hand, it is possible to calculate a rigorous statistical correction to the results obtained in that way.

To facilitate computation we shall assume a Maxwellian distribution of

velocities, and a normal error distribution for the absolute magnitudes. Let these functions be represented by $f(v)dv$ with the constant t and by $\Phi(M)\ dM$ with a value M_o and a modulus of precision h, respectively. Measuring velocities in km./sec., proper motions in seconds of arc per annum, and absolute magnitudes on the international scale, and by using the well known relations between distance r, apparent and absolute magnitude, linear and angular velocity, we find for the law of angular τ components:

$$\chi(\tau)d\tau = 4.74\int_{-\infty}^{+\infty} \Phi(M)dM.f(4.74\tau r).rd\tau$$

$$= 4.74d\tau\int_{-\infty}^{+\infty} \frac{ht}{\pi}\, exp[-h^2(M-M_o)^2-t^2(4.74\,r\tau)^2].rdM \qquad (1)$$

Writing:

$$5\,log\,R=m_o-M_o+5, \qquad r/R=u, \qquad 4.74tR=p, \qquad 5h=k \qquad (2)$$

and substituting in (1), we get:

$$\chi\,(\tau)d\tau = \frac{pk\ mod}{\pi}\,d\tau\int_0^{\infty} exp\,[-k^2\,log^2\,u-p^2\,\tau^2u^2]du \qquad (3)$$

The arithmetic mean value τ_k is given by:

$$\tau_k=\int_0^{\infty}\tau\chi(\tau)d\tau = \frac{1}{4.74\,Rt\sqrt{\pi}}\,10^{1/4k^2\ mod}$$

The "true" arithmetic mean τ component, τ_o, in the case of no dispersion in absolute magnitude is

$$\tau_o = 1/4.74\,Rt\sqrt{\pi}$$

so that we have:

$$\tau_k = \tau_o.10^{1/4k^2\ mod} = \tau_o.10^{\epsilon^2/21.7}$$

if ϵ is the standard deviation of the absolute magnitude distribution. Similarly we find for the relation between the dispersion σ_k in the observed, reduced τ components, and the true dispersion σ_o

$$\sigma_k = \sigma_o.10^{1/2k^2\ mod}$$

Applying the same considerations to the v components we find for the observed law of angular velocities at a distance λ from the apex of solar motion:

$$f(v)dv = \int_o^{\infty} \frac{4.74Rkt\,mod}{\pi}\, exp[-k^2\,log^2\,u-t^2(4.74Rvu-V_o)^2]dudv.$$

if $V_o = Vsin\,\lambda$ and V is the speed of the solar motion.
The algebraic mean value v_k is determined by:

$$v_k = \int_{-\infty}^{+\infty} f(v)v.dv = \frac{V_o}{4.74\,R}\,10^{1/4k^2\ mod} = v_o\,10^{1/4k^2\ mod}$$

if v_o is the "true" value of the v component in this particular area of the sky. As the correction factor is independent of λ, it follows immediately that the same factor applies to reduce the observed secular parallax ρ_k to the "true" value ρ_o.

This factor ρ_k/ρ_o, giving the values of both v_k/v_o and τ_k/τ_o is tabulated in column 2 of table I for different values of $\epsilon = 1/k\sqrt{2}$, the dispersion in absolute magnitude. The third column gives the values of σ_k/σ_o whereas the fourth column exhibits the ratio σ_k/τ_k, and as such indicates the departure of $\chi(\tau)$ from a normal error curve.

TABLE I

ϵ	ρ_k/ρ_o	σ_k/σ_o	σ_k/τ_k
0.0	1.00	1.00	1.25
0.5	1.03	1.06	1.29
1.0	1.11	1.23	1.39
1.5	1.27	1.61	1.58
2.0	1.53	2.34	1.91
2.5	1.94	3.88	2.42

If all the assumptions made were rigorously fulfilled, the value of σ_k/τ_k would afford a good determination of ϵ. In practice, however, we invariably encounter an excess of large deviations in both our normal error curves; accordingly σ_k/τ_k will then give us a value for ϵ which in all probability is a maximum.

When the available material is especially abundant and temptingly homogeneous in nature, it may be possible to use yet a third procedure for the calculation of a mean absolute magnitude. After having found the secular parallax for a group of stars we can correct the individual v components. The residual motions, hereafter referred to as ω components are then more or less comparable to the τ components. The nature of the statistical corrections required to reduce the arithmetic mean ω_k to the "true" mean τ_o component, is found in the following way.

By definition we have:

$$\omega_k = \int_{v_k}^{\infty} f(v)\ (v - v_k)\ dv + \int_{-\infty}^{v_k} f(v)\ (v_k - v) dv$$

Denoting by c the factor $10^{1/4k^2\ mod}$, and writing $V_o t(cu-1) = w$, we obtain:

$$\omega_k = \tau_o \int_0^{\infty} \frac{k\ mod}{\sqrt{\pi}} \frac{du}{u^2}\ exp[-k^2\ log^2 u]\ .\ [|w|\sqrt{\pi}\ erf\ w + e^{-w^2}]$$

At the apex where $V_o = 0$, $w = 0$, and we naturally fall back on the same formula as for τ_k. As soon as V_o, k and t are known, the numerical value of ω_k can be easily ascertained. To get an idea of the magnitude of the factor ω_k/τ_o, the value of the integral has been calculated with data valid for the M giants. With $k = 3.5$, $t = 0.035$, $V = 23$ km./sec. and $\lambda = 90°$,

the value of the integral is 1.26, and the mean value over the whole sky, i.e., over all values of λ, is 1.20. An absolute magnitude computed by comparing the observed value ω_k with the arithmetic mean value of the radial velocity would accordingly require a correction of —0.4.

A possible relation between $\Phi(M)$ and $f(v)$ in the sense of a dependency of luminosity on linear speed, has not been considered above. Any influence exerted by this cause will doubtlessly be small; furthermore the procedure of calculating mean absolute magnitudes in this way is practically confined to giant stars where such a dependency of speed on luminosity seems questionable.

A more important source of error is the neglect of errors of observation and their widely different influence when all stars are reduced to a common apparent magnitude. Owing to the variety of possibilities in this case an adequate statistical correction will have to be derived in almost every case individually.

THE FREQUENCY DISTRIBUTION ON APPARENT MAGNITUDE OF THE NON-MAGELLANIC O-TYPE STARS

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HARVARD SCHOOL OF PUBLIC HEALTH AND HARVARD COLLEGE OBSERVATORY

Read before the Academy November 11, 1924

The class O stars are especially interesting because they are believed to be exceptional from the current point of view of stellar evolution. On the Lane-Russell theory only the most massive stars would reach this class. Eddington's theory would allow also the existence of less massive and very dense class O stars.

Apart from the spectral class, the only property which we can measure for all class O stars is the apparent magnitude. It is interesting to see what this shows us before we go to questions of proper motion (few being well determined), radial velocity, and distance (none being accurately measured) and other hypothetical attributes.

A manuscript catalog compiled at Harvard contains 140 non-Magellanic O stars, including both absorption and emission line stars. If the increase in brightness due to the appearance of bright lines is not significant compared with the large uncertainties (one-half magnitude) in the individual magnitudes of the fainter O stars, we may treat the material as homogeneous. The distribution of apparent magnitude is shown in table I, column 2.

It is practically certain that no emission O star brighter than 8.75 has been overlooked, and it even appears probable that very few emission stars brighter than 10.5 have escaped detection. Assuming the same to hold

also for the absorption O stars we should feel tempted to conclude that the actual numbers of apparently faint O stars are decreasing beyond the ninth magnitude. Closer inspection of table I suggests that a total number of less than 200 stars with apparent magnitudes distributed in a normal

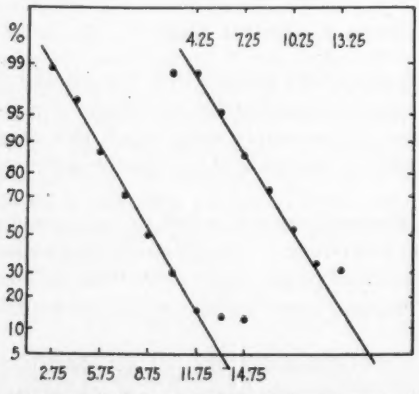
TABLE I

NUMBER OF STARS

APP. MAG.	TOTAL	EMISSION	ABSORPTION
]2.7	2	2	0
2.8– 4.2	3	0	3
4.3– 5.7	16	4	12
5.8– 7.2	29	12	17
7.3– 8.7	30	18	12
8.8–10.2	31	25 *	6
10.3–11.7	24 *	24 (+2)	
11.8–13.2	4 (+13)	4 (+18)	
[13.2	1 (+6)	1 (+12)	
	140 (+19 +1)	90 (+32 +8)	50

* Additional stars in these classes necessary to make the frequency distribution normal.

error curve, may explain the observed distribution very well. The best fit is obtained by assuming a total number of 160, and by taking the constants of the error curve to be: mean $m_0 = 8.5$, dispersion $\sigma = 2.5$ (figure).



Percentage of O stars above assigned apparent magnitudes. (*a*) Total, left hand line, magnitude scale beneath. (*b*) Emission stars only, right hand line, magnitude scale above.

If in spite of the paucity of the material we divide the stars into emission and absorption line stars, and consider each separately, we obtain a somewhat different result. The numbers of emission and absorption line stars for every magnitude interval are given in table I, columns 3 and 4, respectively. We may still regard the numbers in column 3 (emission) as essentially complete as far as 10.5, and on the assumption of a normal distribution predict a total number of 130 emission line O stars whose apparent magnitudes are normally distributed around a mean value of 10.2 with a dispersion of 2.6 (figure). This would make the total number larger than 160.

Many stars classed as B at Harvard are called O at Victoria, and, even

though we allow a possible systematic difference in criteria of classification, we must admit the possibility that some absorption line O stars are missing from the Harvard list among the brighter stars. Accordingly we are not able to say anything very definite about the frequency curve of their apparent magnitudes.

We found a dispersion of 2.6 among the apparent magnitudes of the emission O stars. The factors which may have contributed to produce this dispersion are: dispersion in distance, in temperature and in radius. The latter two combined give us the dispersion in intrinsic brightness. Choosing only the first alternative we calculate a dispersion of 1.2 in the logarithm of the distance, in other words $^1/_6$ of the stars further away than 15 times the average geometric mean distance and $^1/_6$ closer than $^1/_{15}$ that distance. With a mean apparent magnitude of 10.2 and an assumed absolute magnitude of —2, these distances become 40,000 and 200 parsecs. It is practically certain that no O star is so near as 200 parsecs; a distance of the order of 40,000 parsecs would place some of the O stars as far out as the globular clusters.

LIST OF 140 NON-MAGELLANIC CLASS O-STARS

HOURS	MIN.	DEG.	MIN.	MAG.		HOURS	MIN.	DEG.	MIN.	MAG.	
0	37.5	+64	14	10.2	Ob	7	31.4	+17	07	7.7	Oe5
	59.2	+59	53	10.1	Oa		37.4	−32	24	7.7	Oe5
1	32.4	+57	39	10.5	Oc		41.1	−31	41	10.0	Oa
2	16.5	+56	08	9.7	Oe5		42.0	−34	05	11.3	Oa
	19.5	+58	25	8.0	Oe		42.3	−36	16	8.4	Oe5
	33.9	+56	18	10.0	Oa		55.7	−28	28	11.4	Oc
	35.5	+56	28	8.4	Oe	8	00.1	−39	43	2.3	Od
	44.8	+56	31	10.2	Oa		06.5	−47	03	2.2	Oap
3	48.1	+52	21	6.7	Oe5		10.3	−36	38	6.9	Oe5
	52.5	+35	30	4.0	Oe5		35.5	−40	04	7.2	Oe5
5	14.0	+37	20	6.7	Oe5		51.6	−47	13	9.0	Oa
	29.6	+	9 52	3.7	Oe5	9	09.8	−49	42	10.9	Oa
	29.6	+	9 52	5.6	Oe5		51.6	−57	15	8.5	Oc
	30.4	−	5 27	7.9	Oe5	10	07.2	−60	09	10.3	Oa
	30.4	−	5 27	5.4	Oe5		13.5	−57	25	11.4	Oa
	30.5	−	5 59	2.9	Oe5		22.9	−58	08	10.1	Oa
6	03.7	+20	31	7.4	Oe5		31.0	−57	39	8.6	Oe5
	20.8	+	8 03	9.6	Oe		36.0	−60	24	9.4	Oe5
	21.6	+14	57	7.1	Oe5		37.2	−58	03	9.7	Oe5
	35.5	+	9 59	4.7	Oe5		37.4	−59	09	6.5	Ocp
	45.0	−44	13	8.6	Oe5		37.8	−58	15	9.0	Oa
	50.0	−	3 34	9.2	Oe5		40.1	−59	36	6.7	Ocp
	50.0	−23	48	6.6	Ob		40.3	−59	12	8.4	Oc
7	00.8	−	8 34	7.8	Oe5		44.8	−59	42	7.4	Oe5
	04.6	−10	11	6.2	Oe5		47.9	−61	46	12.0	Oa
	13.9	−13	03	12.1	Ob		49.7	−58	59	10.9	Oa
	14.5	−24	23	4.9	Oe		50.5	−58	16	9.4	Oe5
	14.5	−24	47	4.4	Oe5		55.8	−57	17	11.5	Oa

LIST OF 140 NON-MAGELLANIC CLASS O-STARS (*Continued*)

HOURS	MIN.	DEG. MIN.	MAG.		HOURS	MIN.	DEG. MIN.	MAG.	
11	02.3	−64 58	7.8	Oc	17	56.3	−23 01	6.9	Oe5
	05.8	−60 26	8.1	Oa		57.7	−24 22	5.9	Oe5
	10.8	−60 43	8.8	O		59.0	−24 24	6.8	Oe5
12	00.2	−61 29	10.4	Ocp	18	02.1	−19 25	9.8	Ob
	38.0	−62 32	11.0	Oa		02.5	−21 16	7.8	Oa
	50.1	−56 17	5.6	Oe5		07.2	−42 53	12.4	Ocp
13	01.7	−64 46	5.6	Oap		11.0	−16 33	8.7	Oe5
	12.2	−57 37	9.3	Oa		11.6	−18 30	6.4	Oe5
	24.2	−61 34	11.2	Oa		13.5	−11 40	8.9	Oa
	26.8	−61 48	11.2	Oc		17.5	−13 46	10.8	Oa
	27.5	−61 54	10.4	Oe		52.3	−20 33	6.7	Oe5
	35.9	−66 54	9.7	Oa		58.7	− 4 28	11.1	Oc
	44.8	−58 03 Q	8.2	Oe5	19	30.8	+30 18	10.0	Ocp
	48.8	−60 40	11.0	Oa		42.2	+28 01	10.0	Oa
14	07.7	−61 14	6.5	Oe5		44.1	+17 57	11.0	Oa
15	08.9	−60 35	5.2	Oe5		47.9	+18 25	6.3	Oe
	10.8	−60 08	5.5	Oe5		57.8	+32 18	11.1	Oa
	15.8	−62 19	9.4	Obp		59.8	+35 45	6.7	Oe
	21.8	−58 14	10.7	Oe	20	02.2	+35 31	7.0	Op
	55.0	−62 24	9.5	Ob		06.5	+35 53	7.8	Ob
16	16.8	−51 18	11.6	Ob		07.1	+11 35	11.6	Oa
	33.9	−48 34	7.1	Oe5		08.1	+35 54	7.9	Oa
	33.9	−48 34	5.9	Oe5		08.4	+38 03	7.4	Ob
	39.2	−46 55	7.6	Oe		10.8	+36 21	7.9	Oa
	44.5	−41 04	5.4	Oe		13.3	+37 07	8.0	Oc
	45.3	−41 41	6.6	Oc		15.8	+38 25	8.0	Ob
	47.3	−41 40	6.7	Oa		17.1	+43 32	6.8	Oa
	47.9	−44 50	8.1	O		17.8	+36 36	9.4	Oa
	48.0	−41 00	6.0	Oe		24.7	+38 17	13.	Oa
	57.2	−37 42	6.7	Od		49.7	+43 01	9.0	Oe5
17	11.8	−34 18	9.4	Oa		53.1	+44 33	6.0	Oe5
	12.1	−45 32	7.2	Oa	21	14.8	+43 31	5.1	Oe5
	18.2	−43 24	11.6	Oa		35.9	+57 02	5.6	Oe5
	18.5	−34 06	11.8	Oa	22	02.1	+61 48	5.2	Oe5
	26.5	−33 33	11.1	Ob		08.1	+58 56	5.2	Od
	28.1	−32 31	5.7	Oe5		12.9	+55 07	11.1	Oc
	35.3	−33 27	6.7	Oe5		15.0	+55 37	9.0	Ob
	44.8	− 6 40	10.7	Ocp		23.7	+55 46	11.0	Oa
	49.7	−32 27	6.6	Oe5		32.9	+56 23	8.9	Ob
	51.1	−31 00	7.9	Oe5		34.8	+38 32	4.9	Oe5
	52.7	−36 00	7.3	Oe	23	10.8	+59 55	9.2	Oa
	55.2	−32 43	8.8	Obp		28.8	+48 16	[8.3]	Ocp

The second alternative, combined with the assumption of black body radiation would give us a dispersion of 0.26 in the logarithm of the temperature, which would make 10% of the stars hotter than twice the average temperature and 10% cooler than half the average temperature. This range in temperature seems rather excessive as the theoretical values for O star temperatures range from 20,000° to 30,000°. To put all the disper-

sion upon the radius would leave 10% with more than 4 times the average radius and 10% with less than one fourth of that value. Assuming the existence of Eddington's high density stars, the last alternative of large dispersion in radius presents no theoretical difficulties, the only practical objection is that not one of such small, high-density O stars has been found.

Allowing for the limitations of a discussion like the present, it appears reasonably certain that among the O stars occurs a fairly large dispersion in intrinsic brightness and a fairly large scattering through the depths of space. A decision, however, cannot be reached until we have reliable proper motions for the very faint O stars, or until we have, under the additional assumption that the same physical conditions prevail in the Magellanic Clouds as in the Galactic System, definite information about the absence or presence in the Magellanic Clouds of O stars of the twentieth magnitude and fainter.

THE POPULATION OF NEW YORK CITY AND ITS ENVIRONS

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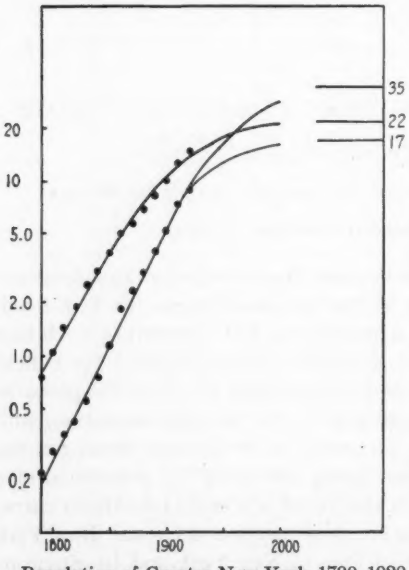
Four years and a half ago Mr. Raymond Pearl read before this Academy the first of the now famous papers by him and his colleague, Mr. L. J. Reed, on following the past course of a population and forecasting its future growth by fitting the curve $1/P = A + Be^{-nt}$ of autocatalytic (or buffer) reaction to the past census counts and extrapolating to predict the probable future course of the population's growth.[1] The two most ambitious subsequent papers are perhaps that presented at the general meeting of the American Philosophical Society last spring discussing the population of a great many countries,[2] albeit with the aid of a generalized type of curve, and the elaborate discussion of the future population of Greater New York City prepared for the Committee of Plan of New York and Its Environs and dealing with many questions of detail significant for long range city planning.[3] The problem of estimating the future population of the Metropolitan district of Boston in connection with planning for an adequate water supply was brought to us at the Harvard School of Public Health some time ago, and at once recourse was had to the Pearl-Reed method as the only one which makes real pretention to scientific forecasting as against merely personal guess-work. On investigating this problem a number of points seemed to thrust themselves forward for consideration in connection with the general method and they may for the present purpose be better

illustrated by commenting on the results already published for New York City than by giving figures for Boston.

The total area which is included for regional planning by the Committee on Plan of New York and Its Environs and for which population predictions are made embraces over 5500 square miles; it is part of the three states, New York, New Jersey and Connecticut, and is, so to speak, New York City proper and its chief commutation areas. This region had in 1920 a population of 9 millions which is 60% of the population (15 millions) of the three states and $8^1/_2\%$ of the total population of the United States. The predicted ultimate (asymptotic) population of the region is 35 millions or $17^1/_2\%$ of that predicted by the same method by these authors for the United States[1] (197 millions). What may be the political, economic, or social consequences of such a concentration of population in and immediately around New York City, or the influence on the public health, and what the system of transportation and distribution whether of persons or of the necessary goods, are questions to intrigue and tax the imagination. Certainly if there is to be such crowding it is well that present and future plans for the city shall take it into consideration long in advance, and should further consider all accessory information available.[4]



Population of Greater New York 1790–1920 (lower curve) with two extensions after 1920 corresponding to the Pearl-Reed and the Wilson-Luyten fittings of the curve, asymptotic respectively at 35 and 17 millions. Population of New York, New Jersey and Connecticut 1790–1920 and extended, asymptotic at 22 millions (upper curve).

Among the items of interest is the condition immediately surrounding the greater city, say, in the three states in which it lies. What is to be their population? Figures for the populations of New York, New Jersey and Connecticut at each decennium from 1790 to 1920 are available in the Census Reports. A curve of the type $1/P = A + Be^{-nt}$ may be fitted to the data, namely, $1/P = 0.045 + 1.264\ e^{-0.0302t}$, where t is measured from 1790 in years. Judged by its percentage departure from the recorded census figures the fit is not bad.[5] In figure 1 are given the census

data for the states and the fitted curve carried on into the future, and the same for the region. The Pearl-Reed limiting population for the region is about 35 millions whereas our limiting value for the three states containing it is a little over 22 millions. The curves cross just before the year 1960. If we follow the curves for 40 years after the recent census, we have the population of the part exceeding that of the whole. As the three states outside the confines of New York City and its environs contained 6 millions in 1920 and the curve of this exterior population shows no signs of falling off (but apparently still tends to increase), the essential conflict between the two curves of growth is emphasized.

This conflict is not surprising. The authors specifically state that the curve can be trusted to forecast only so long as growth remains in the same cycle and no new factors or forces intervene;[6] in their treatment of Germany they break the growth into two cycles. Insofar, therefore, as New York City and its environs on the one hand and the group of the three states of New York, New Jersey and Connecticut on the other hand may be subject to treatment by the simple Pearl-Reed law of population growth and insofar as the curves which they and we, respectively, have fitted may be trusted adequately to represent the growth, we may say that one or the other or both of these populations must very soon change their cycle of growth, if in fact they have not already done so.[7]

The conflict, possibly only to a limited extent so far as practical results go, is logically inherent in the method employed. Populations in different regions are necessarily additive to form the population of the whole. The older types of curve fitted to population growth were additive[8] but had the serious biological defect of giving infinite populations at infinite time and thus failing to predict even qualitatively the inevitable saturation of a region with population. The Pearl-Reed curves, both the simplest one here used and the more complicated ones, satisfy (sometimes[9]) the biological condition for ultimate saturation but fail to satisfy the logical condition of additivity and hence when applied to different parts of a population may well make the limiting population of a part greater than that of the whole.[10] One may as well disregard a logical as a biologic requirement; but in either case the validity of the law established is subject to question, and the use of the law for the prediction is more than ordinarily hazardous.

Although the logical conflict would remain, it seemed worth while, on account of the apparently systematic departures of the populations from the fitted curves, to try a fit by the generalized skew growth curve suggested by the authors and applied by them to Germany and Japan.[11] The first step in making the fit is to determine the limiting population by equation xxvii of p. 577 of *Studies in Human Biology*. Using the enumerated census of New York City and its environs as given by Pearl and Reed[3] at

the five equidistant censuses of 1790, 1820, 1850, 1880 and 1910, an equation was obtained which had no root in the neighborhood of 35 millions, their predicted asymptotic population, and it thus appeared that their general curve could not in this case even approximately reproduce the future results given by the simple curve.[12]

It was a succession of disappointments of this sort, an unexpected instability of the forecasts as obtained by different systems of weighting and by different types of the Pearl-Reed curves that led us to place little confidence in our own ability to apply their method to predict satisfactorily the future population of Greater Boston which must depend for its water supply upon the foresight of Metropolitan District Commission. It was for the purpose of getting light on the method that the examination of the figures for New York was made, and the results of that examination, partly detailed above, leave us skeptical of the forecasts for New York. It is to be hoped that further developments, applications and explanations by the authors will serve to allay our doubts which at present seem shared by a great many economists.

One further difficulty of a mathematical nature may be mentioned. As the curves are not additive the sum of the squares of the residuals, which is the quantity to be minimized in a least squares solution, is not quadratic in the variables (the parameters of the curve), and the derivatives are not simple linear, but complicated transcendental, equations in those variables. It is therefore practically hopeless to work analytically and becomes necessary to explore the whole ground by tedious arithmetic trials without any assurance that there may not be several minima instead of a single value. Indeed when some of the errors are so large as they must be in any non-sinuous curve fitted to the New York population one might expect that the solution would not be unique.

This may be illustrated as follows: The general curve with which we are working is $1/(P - d) = A + Be^{-nt}$ containing four constants. As we shall estimate fit by percentages[4] and as the departures from the Malthusian law are not great the first step is to plot that law, $-\log_{10} P = -0.4343nt \log_{10} B$ – a straight line when P is plotted against t on semi-log paper. The closest fit for New York City and Environs is $\log P = 1.141 + 0.1297t$, where t is measured in decades from the middle year 1855, and the sum of the squares of the residuals is $\sigma^2 = 0.0016$ corresponding to $\sigma = 0.04$ and a "probable error" of 62/3%. The fit is very good,[13] it is better than that given by Pearl and Reed,[3] for which $\sigma^2 = 0.0020$. If we neglect d but now introduce A, we fit[14] $\log (1/P - A) = -0.4343nt + \log B$. For $A = 0.029$ which corresponds to a limiting population of 34.5 millions, we find $\log (1/P - 0.029) = -0.182 - 0.1425t$ with $\sigma^2 = 0.0020$, worse than with $A = 0$. The best fit is very near to $A = 0$ and appears even to be clearly on the negative side (A about -0.01), which would check with our difficulty

in finding an asymptote for the 5-constant solution.[12,13] Next placing A again equal to 0, we may vary d. For $d = 50$ we find $-\log (P - 50) = 1.110 + 0.1369t$ with $\sigma^2 = 0.0015$ which is better.

There is, however, another method of attack. We have the Pearl-Reed estimates of 197 millions for the whole United States.[1] If Greater New York should maintain its relative position with respect to the rest of the country, its limiting population would be around 17 millions. This would not be in conflict with our fitted curve giving 22 millions for the three states, New York, New Jersey and Connecticut, particularly as the indications are that the states may be entering on a period of somewhat accelerated growth. Let us therefore assume for the City a limiting figure of $A = 0.06$ corresponding to $16^2/_3$ millions. Let us choose $d = 120$. Then $\log [1/(P-120)-0.06] = -0.1352-0.1720t$, where t is measured in decades from 1855, and $\sigma^2 = 0.0013$. This is the best fit yet.[15] How much better can be done by suitable variations of d and A we cannot say; the process is tedious. We believe, however, that it is safe to say that the Pearl-Reed method applied with the least squares criterion based on percentages will give a solution for the limiting population much lower than that obtained by them, and correspondingly better suited to our present conceptions of what is economically reasonable and logically not too inconsistent. Such a result is fortunate; for their curves are the best we have, even if, as applied to New York City, there seems to be a great deal of indetermination or divergence in the forecasts.[16]

[1] Pearl, R. and Reed, L. J., these PROCEEDINGS, **6**, 275 (1920). Reprinted in Pearl's Studies in Human Biology, Chap. xxiv.

[2] Pearl, R., *Proc. Amer. Phil. Soc., Philadelphia*, **63**, 10–17 (1924), and studies in Human Biology, Chap. xxv.

[3] Predicted Growth of Population of New York and Its Environs, 130 East 22nd St., New York City, 1923. Not reprinted in Pearl's Studies in Human Biology.

[4] The predictions of N. P. Lewis, appended to those of Pearl and Reed exceed their figures by 25% in the year 2000, only 75 years hence.

[5] The mean percentage error is about 6% as against about 8% for the Pearl-Reed fit to New York and Environs. Neither their curve nor ours is a least square solution based either upon the percentage or upon the actual errors. When the populations vary 20 to 40 fold as here there may be considerable difference between solutions based upon different least square criteria. The authors seem now to prefer criteria based on percentages, Studies in Human Biology, p. 582, although apparently they used actual differences in the New York paper. Furthermore, as their populations are additive at the asymptotes, it is probable that this condition of additivity was in some way especially introduced as limitation to the least squares fitting of the four individual populations.

[6] Studies in Human Biology, pp. 587, 606.

[7] The departures from both curves are in fact systematic. In the states the enumerated populations are too high from 1800 to 1840 and too low from 1870 to 1900; in the City the concavity of the curve from 1790 to 1860 is uniformly wrong, and also from 1870 to 1910. The curve $y = \log P = -\log (A + Be^{-nt})$ has the sign of its second derivative d^2y/dt^2 determined by $-Ae^{-nt}$ and is concave down when $A > 0$, concave, up when $A < 0$. Negative values of A mean that the population is growing on a curve which

becomes infinite in finite time. For these three states the concavity is down from 1790 to 1850, but up from 1860 to 1910; what the process of fitting our curve really does is to average in some obscure fashion, the different concavities so that the net result is a downward concavity. It is worth noting that although the concavity in the data for the City is chiefly upward, both early and late, the fit given by Pearl and Reed actually gives a small net downward concavity and hence a finite though very large limiting population. If we should weight heavily the later populations in the three states we might raise our limiting population of 22 millions almost if not quite indefinitely but the fit in the earlier period would become very bad when judged on a percentage basis.

[8] We refer to the parabolic and logarithmic parabolic curves mentioned by Pearl and Reed, these PROCEEDINGS, **6**, 1920, p. 275. The Malthusian curve $P = ce^{kt}$ is of course non-additive.

[9] Any of the Pearl-Reed curves fitted to actual enumerations may bring about such a determination of their constants as to make the population infinite in finite time instead of infinite only at infinite time as was the case with the earlier types of additive curves (or the Malthusian curve) that were used. See note 6 and E. B. Wilson, *Science* **61**, 1925 (87).

[10] A simple fictitious example may show this. Suppose that at three equidistant censuses, not necessarily consecutive, the populations of one region are in millions or other units respectively 1, 3, 8 and of another are 1, 4, 9, so that the population of the total region is 2, 7, 17. Then the limiting populations are 33, 12.57 and 30.33 which makes the population of the whole less than that of one part although that of the other part is considerable. The summation of autocatalytic curves has been carefully studied at the Harvard Medical School because of its importance for the theory of the protein titration curve, see E. J. Cohn, *Proc. 11th Internat. Physiol. Congress*, Edinburgh (1923), p. 91.

[11] Studies in Human Biology, pp. 572–577, 607, 625.

[12] The same appears to be true if the data for 1800, 1830, 1860, 1890, 1920 are used. An equation of effectively odd degree like their xxvii must have at least one real root and their five constant equations can be fitted to either set of data, but the solution gives an infinite population in finite time.

[13] The Malthusian curve does in fact fit so very well that one may seriously question whether there is any statistical justification for seeking improvement. If we try a parabola $\log P = a + bt + ct^2$ we find $c = +0.00056$, a net upward concavity (see notes 9 and 12), but only equal to its probable error.

[14] The weights for the percentages are no longer quite equal because $\log (1/P - A)$ replaces $\log 1/P$ and the quantity of equal weight is $\log [(1 - AP)/P]$, but the difference is not great.

[15] Whether this last fit is significantly better than any of the others may be doubted. With $\sigma = 0.04$ and 14 data, the standard deviation of σ is by the usual formula $\sigma/\sqrt{2N}$ $0.04/\sqrt{28} = 0.008$ which covers the entire range from 0.036 just found to the Pearl-Reed value 0.045 with considerable margin to spare. The values for the census 1790–1920 for our curve and for the Pearl-Reed curve are given below, and the forecasts from 1930–2000, including that of Lewis.

	1790	1800	1810	1820	1830	1840	1850	1860	1870	1880	1890	1900	1910	1920
Census	225	290	359	428	560	764	1163	1835	2375	3026	3966	5385	7467	8979
W and L	223	273	347	456	613	839	1169	1633	2271	3141	4235	5584	7113	8755
P and R	184	254	350	482	662	908	1242	1692	2295	3095	4131	5460	7117	9122

	1930	1940	1950	1960	1970	1980	1990	2000	Asymptote
W and L	10370	11840	13110	14130	14890	15460	15870	16160	16667
P and R	11460	14070	16840	19650	22340	24810	26960	28760	34900
Lewis	11000	13800	16600	20000	24000	28000	32000	36000	...

We believe that the Pearl-Reed values are very near to those obtained by basing the least squares criterion on actual instead of percentage differences. If so, the system of weighing determines almost completely the solution obtained.

[16] If it were not for the indeterminateness we should be interested in the fact that for $d = 120$ and $A = 0.06$, the logarithmic errors are not only small but non-systematic and in fact oscillating with an amplitude of about 0.045 and period of about 50 years. A great commercial city like New York might easily show an oscillation of about 10% in population about any law of growth during such major economic cycles as have occurred during the past century with about a fifty-year period (prices high in 1815, low in 1845, high in 1865, low in 1897, high in 1919). The population of the city appears to lag behind the growth curve in extended periods of liquidation and to shoot ahead of it in periods of inflation; but one should not generalize on such meager and speculative material.

ON THE INTERSECTION INVARIANTS OF A MANIFOLD

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Consider a manifold M that can be subdivided into a finite number of n-cells, bounded and separated from one another by a finite number of cells of lower dimensionalities. If we denote the sensed i-cells of any subdivision S_0 of M by

$$[i]_1, [i]_2, \ldots, [i]_{\alpha i} \tag{1}$$

respectively, the relation of an i-cell $[i]_s$ to the $(i-1)$-cell on its boundary may be indicated by writing

$$[i]_s \equiv [_{i-1}{}^{i}]_s^t [i-1]_t, \tag{2}$$

(summed for t), where $[_{i-1}{}^{i}]_s^t$ is a coefficient which vanishes if $[i-1]_t$ is not on the boundary of $[i]_s$ but is otherwise equal to $+1$ or -1 according as $[i-1]_t$ is positively or negatively incident to $[i]_s$.

An *i-chain* is any linear combination $\lambda^s[i]_s$ of i-cells $[i]_s$, where the coefficients λ^s are integers. Its boundary is the right-hand member of the congruence

$$\lambda^s [i]_s \equiv \lambda^s [_{i-1}{}^{i}]_s^t [i-1]_t \tag{3}$$

derived by combining congruences of the type (2). If the boundary of an i-chain vanishes, the i-chain is *closed*. If an i-chain $(i)_0$ is the boundary of an $(i+1)$-chain, the i-chain is *bounding*, or *homologous to zero:* A set

$$(i)_0 \sim 0$$

of closed i-chains is *linearly independent with respect to bounding* if no linear combination of them is bounding. We make the convention (different

from the one in Veblen's Colloquium Lectures) that every 0-cell $[0]_s$ is a closed chain.

Let P_i be the maximum number of closed i-chains, independent with respect to bounding, that may be formed from the α_i i-cells of S_0. The numbers P_i satisfy the well known relation

$$P_i = \alpha_i - p_{i-1}^{\ i} - \rho_i^{i+1} \tag{4}$$

where $p_{i-1}^{\ i}$ is the rank of the matrix of the coefficients $[{}_{i-1}^{\ i}]_s^t$. This formula is valid for $i = 0$ and n if we put $\rho_{-1}^{\ 0} = \rho_n^{n+1} = 0$.

By an easy elimination, (4) leads to

$$\sum_{i=o}^{n} (-1)^i (P_i - \alpha_i) = 0 \tag{5}$$

the so-called Euler-Poincaré formula.*

The coefficients of (2) are known to satisfy the relations

$$[{}_{i-1}^{\ i}]_s^t [{}_{i-2}^{\ i-1}]_t^u = 0 \tag{6}$$

which merely express the fact that the boundaries of the cells $[i]_s$ are closed. Let us introduce a set of symbols

$$[{}_{n}^{n+1}]_s^t = 1$$

If, with the aid of these symbols and by properly assigning senses to the n-cells, we can extend relations (3) to the case $i = n + 1$, the sum of the n-cells of S_0 is a closed n-chain and the manifold M is said to be *orientable.*

Now, let the coefficients of the congruences (2) and (3) be reduced by any integer modulus π. The formulas (4) and (5) are in no way altered, except that $\rho_{i-1}^{\ i}$ must, this time, be interpreted as the rank of the matrix of coefficients modulo π, and P_i as the number of i-chains, closed mod π and independent mod π with respect to bounding. The new numbers P_i may be called the *i-th connectivity numbers of M modulo* π. It is easy to see that a knowledge of the connectivity numbers for all values of the modulus π gives exactly the same information about M as a knowledge of the Betti numbers and coefficients of torsion of Poincaré. However, the modular invariants have the advantage of generalizing, as we shall see in a moment.

If the manifold M is orientable, the modular connectivity numbers P_i satisfy duality relations

$$P_i = P_{n-i}$$

* As written above, this formula presents a somewhat unfamiliar appearance owing to a slight departure from usage in the definition of the numbers P_i.

like those satisfied by the Betti numbers of Poincaré. From a certain value of the modulus π on, the numbers P_i are independent of π. It is, therefore, never necessary to consider more than a finite number of them.

Let us now assume that M is orientable and that the n-cells of S_0 have been sensed in such a manner that (6) is valid for $i = n + 1$. Then, we know that there exists a subdivision S^0 dual to S_0 made up of cells $\{n - i\}^s$ dual to the cells $[i]_s$ of S_0, respectively. The cells of the dual subdivision S^0 are incident as expressed by the congruences

$$\{i\}^s \equiv \{{}^{i}_{i-1}\}^{s}_{t} \{i-1\}^t, \tag{7}$$

where the coefficients of (2) and (7) are related by

$$[{}^{i}_{i-1}]^t_s = \{{}^{n-i+1}_{n-i}\}^t_s.$$

An i-cell $[i]_s$ of S_0 intersects a j-cell $\{j\}^t$ of S^0 either not at all in which case we write

$$[i]_s \{j\}^t = 0,$$

or else in a cell of dimensionality $i + j - n$ which we denote by $[i]_s \{j\}^c$. The cells of the latter type belong to a subdivision S obtained by superimposing S_0 and S^0. In the particular case where $j = n - i$, the symbol $[i]_s \{j\}^t$ denotes a cell if, and only if, $s = t$. With proper conventions about the orientation of the cells of S, we have the two fundamental relations

$$[i]_s \{j\}^t = (-1)^{(n-i)(n-j)} \{j\}^t [i]_s \tag{8}$$

$$[i]_s \{j\}^t \equiv \{{}^{j}_{j-1}\}^t_u [i]_s \{j-1\}^u + (-1)^{n-j} [{}^{i}_{i-1}]^u_s [i-1]_u \{j\}^t \tag{9}$$

Because of this last set of relations, the congruences

$$(i)_0 \equiv (i-1)_0, \qquad (j)^0 \equiv (j-1)^0,$$

where $(i)^0$, etc., are arbitrary chains, imply

$$(i)_0 (j)^0 \equiv (i)_0 (j-1)^0 + (-1)^{n-j} (i-1)_0 (j)^0 \tag{10}$$

We readily conclude from (10) that if the closed chain $(j-1)^0$ is bounding so also is its intersection with an arbitrary closed chain $(i)_0$. This tells us, in turn, that if two closed chains $(i)_0$ and $(j)^0$ are transformed into homologous chains, respectively, their intersection is transformed into an homologous chain.

By (6), (9) and (10), the intersection of two closed chains is a closed chain. The intersection of a chain $(i)_0$, closed mod π_1, with a chain $(j)^0$ closed mod π_2 is a chain $(i)_0 (j)^0$ closed mod π, where π is the H.C.F. of π_1

and π_2. This is seen by expressing the chains $(i)_0$ and $(j)^0$, in terms of their individual cells. The coefficients of $(i)_0$ will be indeterminate to a factor $\lambda_1 \pi_1$ those of $(j)^0$ to a factor $\lambda_2 \pi_2$, and, therefore, those of $(i)_0 (j)^0$ to a factor $\lambda_1 \pi_1 + \lambda_2 \pi_2 = \lambda \pi$.

With these preliminaries indicated, let $(i)_s$ be a complete set of i-chains of S_0, closed and indépendent with respect of bounding mod π_1, $(j)_0$ a similar set of j-chains mod π_2, and $(k)_a$ a similar set of $k = i + j - n$ chains mod π (H.C.F. of π_1, and π_2). We then have

$$(i)_s (j)_t \sim \alpha^u_{st} (k)_u. \tag{11}$$

The actual calculation of $(i)_s (j)_t$ is, of course, made by substituting for $(j)_t$ an homologous chain $(j)^t$ of S^0, calculating the intersection in S of $(j)^t$ with $(i)_s$, and substituting for the chain of intersection $(i)_s (j)^t$ in S an homologous chain $\alpha^u_{st} (k)_u$ of S_0. The form determined by the coefficients α^u_{st} is a characteristic of the manifold M, from which a number of topological invariants may be derived. I have indicated a few of these invariants in three recent notes to these PROCEEDINGS. The general theorem about the forms α^u_{st} is as follows.

A necessary condition that two manifolds M and M' be homeomorphic is that the forms α^u_{st} determined for all possible choices of i, j, π_1, and π_2 be transformable simultaneously into the corresponding forms of M' by unimodular transformations on the sets of variables $(i)_s$, $(j)_t$, $(k)_u$.

In closing, it may be said that so long as we are dealing with a two dimensional matrix, such as the matrix of the coefficients of (2), it is immaterial whether we express the invariants of the matrix in terms of its rank and elementary divisors or in terms of its ranks modd 2, 3, ... etc. However, the notion of rank modulo π generalizes in several obvious ways to matrices of higher dimensions, such as the matrices of the coefficients α^u_{st} of (11). Suppose, for example, the general term of an n-dimensional matrix is $\alpha_{i_1 i_2 \ldots i_n}$. We may divide the subscripts into two groups of i and $n - i$, respectively, separate the terms $\alpha_{i_1 i_2 \ldots i_u}$ into classes such that each class corresponds to a fixed determination of the i subscripts of the first group, and consider the number of linearly independent classes modulo π.

A more detailed and systematic exposition of the above will be given elsewhere.

THE ROTATING DISC[1]

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In connection with the theory of relativity much attention has been given to the problem of what happens, under hypotheses consistent with this theory, when a disc made of homogeneous incompressible material is caused to rotate. Einstein[2] gave a qualitative discussion which indicated that the geometry on the disc could not be considered to remain Euclidean. Quantitative discussions taking account of the cohesive forces in the disc were given later by Lorentz[3] and Eddington.[4] These latter discussions take as their starting point equations of transformation corresponding to a Euclidean rotation, which in the nature of things allow for no contraction, and yet end by computing a contraction. Furthermore, even if the significance of radial distance in the transformation were changed to allow for this, the geometry would necessarily remain Euclidean, in contradiction to the earlier discussion. The reconciliation of these opposed points of view forms the object of this note.

Before rotation, we have in cylindrical coördinates:

$$ds'^2 = dt'^2 - (dr'^2 + r'^2\, d\theta'^2 + dz'^2). \qquad (1)$$

If we assumed that the geometry remained Euclidean, and that the equations of rotation were in the usual form

$$r = r',\ \theta = \theta' + \omega t',\ z = z',\ t = t',$$

the new expression would be:

$$ds^2 = dt^2 - dr^2 - r^2(d\theta - \omega dt)^2 - dz^2. \qquad (2)$$

If we admit the possibility of non-euclidean geometry, but retain the form of this expression in so far as it depends on θ, z and t, we are led to write:

$$ds^2 = dt^2 - dR_1^2 - R_2^2(d\theta - \omega dt)^2 - dz^2, \qquad (3)$$

which we shall use as the basis of our discussion. R_1 and R_2 are functions of r and ω. R_1 may be considered to be the new radial measure of the disc since it is the spatial interval, $\int \sqrt{-ds^2}$, between the origin and a given point, measured along a path for which $d\theta = dz = dt = 0$.

Our interpretation of incompressibility is that the closeness of packing of the molecules is independent of the applied stress system. Consequently

the proper particle density, is unchanged by the rotation. If dW is the proper volume,

$$N = \int \sigma \, dW, \tag{4}$$

the integral being taken over the entire disc, is the total number of particles in the disc and must therefore be independent of the angular velocity, ω.

To determine dW in the new coördinates, we have

$$dW \, ds = \sqrt{-|g_{ij}|} \, dR_1 d\theta \, dz \, dt = R_2 \, dR_1 \, d\theta \, dz \, dt, \tag{5}$$

where we put g_{ij} for the coefficients in (3). But, for an element of the rotating disc, z, r and θ are constant, so that the proper-time is given by

$$ds = \sqrt{1-\omega^2 R_2^2} \, dt, \tag{6}$$

and

$$dW = (1 - \omega^2 R^2)^{-1/2} \, R_2 \, dR_1 \, d\theta \, dz. \tag{7}$$

Thus if the thickness of the disc is b, and its new radius is A_1, we have:

$$N = \int \sigma \, dW = 2\pi \, \sigma b \int_0^{A_1} (1 - \omega^2 R_2^2)^{-1/2} \, R_2 dR_1. \tag{8}$$

If the contraction in radius at a given distance only depends on the angular velocity and the matter inside this distance, we will have

$$A_1 = R_1(a,\omega) \tag{9}$$

where $R_1 = R_1(r, \omega)$ and a is the radius of the disc at rest. Furthermore, since N is independent of ω, it must equal the value given by (8) when $\omega = 0$. In this case, as a comparison of (1) and (3) shows, $R_1 = R_2 = r$; $A_1 = a$, so that we have

$$N = 2\pi \, \sigma b \int_0^a r \, dr = \pi \, \sigma b \, a^2. \tag{10}$$

This shows that $R_1(r, \omega)$ and $R_2(r, \omega)$ must be such that

$$\int_0^{R_1(a,\omega)} (1 - \omega^2 R_2^2)^{-1/2} \, R_2 dR_1 = \tfrac{1}{2} \, a^2. \tag{11}$$

which is equivalent to:

$$(1 - \omega^2 R_2^2)^{-1/2} \, R_2 \, \frac{\partial R_1}{\partial r} = r. \tag{12}$$

To proceed further with the computation, some additional assumptions must be made. If we assume $R_2 = r$, we find:

$$R_1(r, \omega) = \frac{1}{2\omega}\left(sin^{-1}\omega r + \omega r\sqrt{1 - \omega^2 r_2}\right) \tag{13}$$
$$= r - \frac{\omega^2}{6} r^3 - \frac{\omega^4}{40} r^5 \ldots .$$

If we assume $R_1 = r$, we find:

$$R_2(r, \omega) = r(1 + \omega^2 r^2)^{-1/2} = r - \frac{\omega^2}{2} r^3 + \frac{3\omega^4}{8} r^5 \ldots \tag{14}$$

This agrees, up to the term in r^3, with the result obtained by applying the Fitzgerald contraction to the circumference directly.[2]

If we assume $R_1 = R_2$, we find:

$$R_1(r, \omega) = R_2(r, \omega) = r\sqrt{1 - \frac{\omega^2}{4} r^2} \tag{15}$$
$$= r - \frac{\omega^2}{8} r^3 - \frac{\omega^4}{32} r^5 \ldots$$

which agrees with the result found by Lorentz[3] and Eddington.[4]

It is interesting to compute the Gaussian curvature of the surface of the disc ($z = t = 0$) in these three cases. We find

$$k = -\frac{\omega^2}{(1 - \omega^2 r^2)^2}, \tag{13.1}$$

$$k = \frac{3\omega^2}{(1 + \omega^2 r^2)^2}, \tag{14.1}$$

$$k = 0, \tag{15.1}$$

respectively. (13) and (14) stand out as extreme cases, while (15) is intermediate. Some further refinement of the concept of incompressible rotation, and the action of the internal stresses is necessary to distinguish between these various possibilities. Any reasonable theory will probably yield values for the contractions and curvature somewhere between the cases (13) and (14). In view of the remarks in the first paragraph, the right of (15) to preëminence does not seem to have been conclusively proved.

[1] Presented to the American Mathematical Society, October 25, 1924.

[2] Relativity, A Popular Exposition, by Albert Einstein, translated by R. W. Lawson, London, 1920, pp. 79–82.

[3] *Nature*, **106**, p. 795.

[4] The Mathematical Theory of Relativity, by A. S. Eddington, Cambridge, 1923, p. 112.

TRANSITIVE GROUPS INVOLVING DIRECT PRODUCTS OF LOWER DEGREE

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Let G represent a transitive substitution group of degree n and let G_1 represent the group composed of all the substitutions of G which omit a given letter. It will be assumed in what follows that G_1 is intransitive and that it is the direct product of its transitive constituent groups. We shall first consider the case when at least one of these constituent groups of smallest degree is non-regular, and shall represent such a constituent by T. If G_1' represents the group composed of all the substitutions of G which omit a letter of T then it is easy to see that G_1' involves all the letters of G which do not appear in G_1. Moreover, these letters must appear in the constituent of G_1' which corresponds to T in G_1. This constituent must have at least one letter in common with T.

The group generated by G_1 and G_1' must therefore be of degree n and it has the same number of transitive constituents as G_1 has. It is also the direct product of its transitive constituents. Its transitive constituents of lowest degree must be of a larger degree than the degree of T. Otherwise, the group composed of all the substitutions of G which omit a letter of this constituent of lowest degree could not be conjugate with G_1. Hence the following theorem has been established: *If the subgroup composed of all the substitutions which omit a given letter of a transitive group is both intransitive and the direct product of its transitive constituents, and if at least one of its transitive constituents of lowest degree is non-regular, then it contains only one constituent of lowest degree.*

The degree of a constituent of next to the lowest degree in G_1 is equal to the degree of T increased by the number of letters omitted by G_1 since the subgroup composed of all the substitutions of G which omit a letter of such a constituent is conjugate with G_1 under G. In fact, the subgroup generated by G_1 and G_1' is composed of all the substitutions of G which transform among themselves the letters of each of the smallest transitive constituents of this subgroup. If all the transitive constituents of lowest degree in G_1 are regular, or if there is only one such constituent in G_1 and it is regular, then all the constituents of lowest degree in G_1 must be simply isomorphic. If there is more than one such constituent the degree of each of these regular constituents is equal to the number of letters omitted by G_1. If G_1 contains constituents of more than one degree then every constituent of next to the lowest degree must be non-regular and the degree of such a constituent is equal to the sum of the degrees of the con-

stituents of lower degree increased by the number of letters omitted by G_1.

Since the subgroup G_1'' composed of all the substitutions of G which omit a letter of a constituent of the next to the lowest degree contained in G_1 together with G_1 always generates a group which is the direct product of its transitive constituents, and as this direct product is always composed of all the substitutions of G which transform among themselves the letters of the smallest transitive constituents of this group, it results that the direct product of the conjugates of a constituent of next to the lowest degree in G_1 is always an invariant subgroup of G. In particular, it has been proved that *if the subgroup G_1 composed of all the substitutions which omit a given letter of a transitive group is both intransitive and the direct product of its transitive constituents then either all these transitive constituents are non-regular or all the constituents of lowest degree in G_1 are regular while all the other transitive constituents are non-regular.* In the latter case there may be only one transitive constituent of lowest degree on G_1.

It was noted above that the direct product of the conjugates under G of a transitive constituent of next to the lowest degree in G_1 constitute an invariant subject of G. In fact, when a transitive constituent of lowest degree in G_1 is regular and if the degree of this constituent is equal to the number of letters omitted by G_1 then two conjugates of a transitive constituent of lowest degree in G_1 are either identical or they have no letter in common. Hence in this case the direct product of these conjugates is also an invariant subgroup of G. The arguments employed with respect to the conjugates of the constituents of next to the lowest degree in G_1 can evidently be applied, step by step, to constituents of higher degrees, if such constituents appear in G_1. Hence the following theorem: *If G_1 involves transitive constituents of different degrees then the degree of each such constituent is equal to the sum of the degrees of all the constituents of lower degree increased by the number of letters omitted by G_1.* Moreover, the distinct conjugates of any such constituent which is not of lowest degree have no letter in common and the direct product of these conjugates is invariant under G.

As a special case included in the preceding theorems it may be noted that when all the transitive constituents of G_1 are regular then they must be simply isomorphic and the direct product of the conjugates of one of these constituents must be an invariant subgroup of G. The transitive constituents of this invariant subgroup are transformed by G according to a regular substitution group. As this invariant subgroup is also invariant under all the substituents on the letters of G which transform G into itself and as its transitive constituents can be transformed according to any regular group whose order is equal to the number r of these transitive constituents it results that there are at least as many different transitive groups which contain the given G_1 involving r-I regular transitive constit-

uents as there are abstract groups of order r. It is not difficult to prove that this is the actual number of such groups.

To illustrate the remarks of the preceding theorem we may let G_1 represent the intransitive group of degree 4 and of order 4. In this case $r = 3$ and G is the imprimitive group of degree 6 which is of order 24 and contains an invariant subgroup of order 8. Since there is only one abstract group of order 3 there is only one group which contains the given intransitive group as its G_1. It may be added that the given theorems exhibit the fact that comparatively few intransitive substitution groups which are the direct product of their transitive constituents can be used as the G_1 of at least one transitive group. What is more important is the fact that the possible transitive groups which can involve such a group as its G_1 are sometimes completely determined by this subgroup. At any rate, the theorems announced above may serve to exhibit the fact that the method of studying transitive substitutions group more extensively by means of their subgroups composed of all their substitutions which omit a given letter leads to a deeper insight into the structure of some useful substitution groups and their determination.

THE RESPIRATORY FUNCTION OF THE HEMOCYANINS

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Read before the Academy November 12, 1924

The bloods of a large number of only remotely related invertebrates contain compounds of protein which change from a blue color when in the presence of oxygen to an almost colorless condition when all traces of free oxygen are removed. To these substances as a class the name hemocyanin has been applied. Elementary analysis has demonstrated that in a number of organisms the hemocyanins are specifically different chemical compounds.[1]

Our observations on the respiratory function of the hemocyanins of the squid, Loligo peali, and of the horse-shoe crab, Limulus polyphemus, offer new evidence for the specificity of the hemocyanins in these two forms and suggest an interesting correlation between the chemical properties of these respiratory proteins and the functional peculiarities of the animals.

The equilibrium between hemocyanin and oxygen in the blood sera have been studied by a colorimetric method. The blue color of the serum is undoubtedly due to the formation of an oxygen compound of hemocyanin which may be called oxyhemocyanin; in the reduced condition the hemo-

cyanin is colorless. By comparing the color of the serum with known mixtures of oxygenated and reduced hemocyanin, or with artificial color standards made up to match such solutions, an estimate of the relative proportions of oxyhemocyanin and reduced hemocyanin may be made which we consider accurate to ± 5 per cent of the total amount of hemocyanin present.

The conditions of equilibrium between oxygen, hemocyanin and oxyhemocyanin in the sera of Loligo and Limus are defined by the oxygen dissociation curves in figure 1. The data on which this figure is based were obtained by equilibrating small quantities of serum with analyzed mixtures

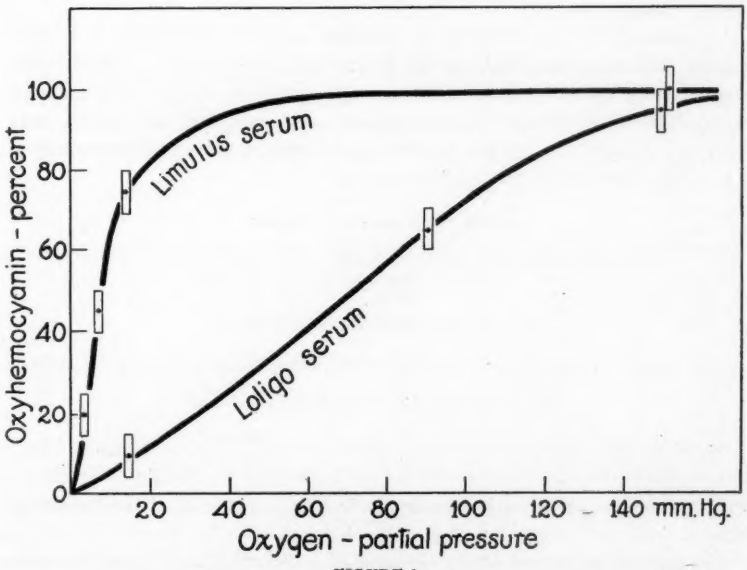

FIGURE 1

Oxygen dissociation curves of serum of Loligo peali and Limulus polyphemus in the virtual absence of CO_2 and at 22°C

of oxygen and nitrogen in which carbon dioxide occurred in traces less than 0.5 per cent. It is clear that under comparable conditions Limulus hemocyanin has a much greater affinity for oxygen than has that of Loligo.

The effect of the presence of carbonic acid upon the equilibrium is demonstrated by comparing the color of small quantities of sera equilibrated with gas mixtures containing equivalent proportions of oxygen but different proportions of carbon dioxide. In the case of Loligo serum the pres-

ence of carbon dioxide decreases the affinity of hemocyanin for oxygen. The magnitude of the effect is indicated by the following data:

LOLIGO PEALI—SERUM

O_2 PARTIAL PRESSURE MM.	CO_2 PARTIAL PRESSURE MM.	OXYHEMOCYANIN PER CENT
93.5	41.2	10 ± 5
91.5	5.0	65 ± 5
14.5	4.0	10 ± 5

In this species the effect of carbon dioxide upon the oxygen dissociation curve is similar to that observed by Bohr, Hasselbalch and Krogh[2] in the case of human blood.

In contrast to the foregoing, in Limulus serum the presence of carbon dioxide increases the affinity of the hemocyanin for oxygen. When equilibrated with equal partial pressures of oxygen the serum is invariably bluer if there is carbon dioxide present, and provided, of course, that there is not sufficient oxygen present to completely saturate the hemocyanin. The following figures are illustrative:

LIMULUS POLYPHEMUS—SERUM

O_2 PARTIAL PRESSURE MM.	CO_2 PARTIAL PRESSURE MM.	OXYHEMOCYANIN PER CENT
4.8	24.2	40 ± 5
3.6	trace	20 ± 5
8.5	84.4	85 ± 5
7.2	2.5	45 ± 5
13.9	70.6	95 ± 5
13.8	2.8	75 ± 5

This is evidence of an entirely new sort for the specificity of the hemocyanins.

The specific properties of the hemocyanins of Loligo and Limulus appear to bear some relation to the functional characteristics of these animals. An oxygen-carrying substance in the blood can only be of physiological significance when the range of oxygen tensions between the lung or gill and the active tissues coincides with a part of the oxygen dissociation curve in which the relative proportions of oxygenated and reduced protein change considerably. This is the case both in Loligo and Limulus, for the color of the blood when drawn from the living animal indicates that its hemocyanin is only partially oxygenated. We have shown that this range of oxygen tensions is much higher in Loligo then in Limulus. Correlated with this is the fact that Loligo is a pelagic form normally exposed only to sea water saturated with air, whereas Limulus, because of its tendency to burrow in the

sand, must frequently be exposed to much lower oxygen tensions. A similar relation between the habitats of fishes and their oxygen dissociation curves has been found by Krogh and Leitch.[3] There appears also to be some relation between these facts and the ability of these animals to withstand a lack of oxygen. Limulus is most resistant to adverse respiratory conditions and Newman[4] has shown that its heart will continue to beat in the absence of oxygen practically as long as in its presence. Loligo, on the other hand, is difficult to keep alive in an aquarium under the best conditions, and dies promptly if the oxygen in the water falls to a third of that normally present in the sea.[5] It is suggested that there may be a relationship between the tensions at which the tissues of an organism utilize oxygen to best advantage and the pressures at which its respiratory proteins can carry on the oxygen exchange.

[1] Alsberg and Clark, *J. Biol. Chem.*, **8**, 1 (1910).
[2] Bohr, Hasselbalch and Krogh, *Skand. Arch. Physiol.*, **16**, 390 (1907).
[3] Krogh and Leitch, *J. Physiol.*, **52**, 288 (1919).
[4] Newman, *Amer. J. Physiol.*, **15**, 371 (1906).
[5] Amberson, Mayerson and Scott, *J. Gen. Physiol.*, **7**, 171 (1924).

PROLONGED FASTING AS AFFECTING THE COMPOSITION OF STEERS' URINE

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In a coöperative study on the metabolism of steers with varying degrees of nutrition, carried on for several years by the Nutrition Laboratory and the New Hampshire Agricultural Experiment Station,[1] two steers have recently been studied during absolute fasts of 5, 7, 10 and 14 days. Since, when fasting, a steer lives upon its own flesh, and the alkaline urine characteristic of the herbivora changes to the acid urine of the carnivora, a study of the changes in the urine under these conditions throws light upon the changes in the fundamental metabolism as the fast progresses. The steers' urines have therefore been analyzed[2] and the results are presented in this communication in tables 1 and 2. The only previous study of the fasting urines of steers with which these results may be compared is that made by Grouven,[3] who determined the hippuric acid and ash in the urines of several days, finding a high amount of phosphate.

The details as to previous diet, period of fast and body-weight of the steers are given in the footnotes to the tables. So far as possible, the urines were collected in 12-hour periods, though irregularity in voiding

TABLE 1

THE CONSTITUENTS OF THE URINE OF STEER C AS AFFECTED BY FASTING.[1]
(Amounts per hour)

DATE	TIME	VOLUME	TOTAL NITROGEN	UREA NITROGEN	AMMONIA NITROGEN	AMINO-ACID NITROGEN	HIPPURIC ACID NITROGEN	PREFORMED CREATININE	TOTAL CREATININE	INORGANIC SULPHATE (S)	ETHEREAL SULPHATE (S)	NEUTRAL SULPHUR (S)	PHENOLS		ACETONE AND DIACETIC ACID	β-OXYBUTYRIC ACID	TOTAL FIXED BASES 0.1 N	ORGANIC ACIDS[2] 0.1 N
													FREE	CONJUGATED				
		CCMS.	GRAMS	GRAMS	GRAMS	GRAMS	GRAMS	GRAM	GRAM	GRAMS	GRAMS	GRAMS	GRAMS	GRAMS	GRAM	GRAMS	CCMS.	CCMS.
1924																		
Mar. 2–3	2.00 P.M.–12.08 P.M.	161	1.49	——	——	0.265	0.409	0.72	0.80	0	0.142	0.010	0.087	0.147	tr.	0.044	810	555
Mar. 3	12.08 P.M.–4.25 P.M.	121	0.99	0.26	0.067	0.059	——	.61	.63	0	.094	.003	.057	.078	tr.	.029	765	285
Mar. 3–4	4.25 P.M.–6.51 A.M.	92	0.79	.32	.017	.024	.708	.54	.59	0	.081	.007	.076	.057	tr.	.025	470	155
Mar. 4	6.51 A.M.–11.40 P.M.	140	1.12	.33	.024	.028	.106	.60	.61	0.066	.026	.014	.137	.022	tr.	.028	410	180
Mar. 4–5	11.40 A.M.–6.50 A.M.	75	1.27	.74	.014	.014	.080	.50	.57	0	.094	.002	.030	.084	tr.	.017	210	135
Mar. 5	6.50 A.M.–10.14 A.M.	——	——	——	——	——	——	——	——	——	——	——	——	——	——	——	——	——
Mar. 5	10.14 A.M.–5.16 P.M.	92	1.46	.93	.047	.017	.053	.54	.69	.021	.040	.009	.071	.027	tr.	.013	120	130
Mar. 5–6	5.16 P.M.–4.02 P.M.	67	1.61	.94	.026	.012	.037	.58	.79	0	.036	0	.020	.053	tr.	.009	80	120
Mar. 6–7	4.02 P.M.–6.58 A.M.	61	1.63	1.06	.032	.013	.031	.56	.79	0	.028	.001	.011	.031	tr.	.010	65	120
Mar. 7	6.58 A.M.–5.09 P.M.	62	1.62	1.01	.060	.009	.029	.57	.65	0	.023	.006	.018	.019	.002	.007	65	120
Mar. 7–8	5.09 P.M.–7.50 A.M.	62	1.61	1.08	.046	.009	.029	.56	.60	0	.012	.020	.009	.029	.003	.007	70	115
Mar. 8–9	7.50 A.M.–7.02 A.M.	65	1.61	1.10	.041	.008	.026	.54	.57	0	.034	.008	.006	.029	tr.	.008	85	110
Mar. 9	7.02 A.M.–4.59 P.M.	64	1.70	1.12	.060	.009	.029	.62	.63	0	.029	.014	.006	.026	tr.	.010	90	125
Mar. 9–10	4.59 P.M.–7.00 A.M.	76	1.68	1.19	.049	.011	.029	.61	.60	.025	.017	.017	.008	.023	.005	.010	125	120
Mar. 10	7.00 A.M.–3.42 P.M.	69	1.60	1.11	.060	.012	.026	.58	.56	.032	.023	0	.007	.022	.002	.010	140	120
Mar. 10–11	3.42 P.M.–7.35 A.M.	68	1.56	1.05	.053	.011	.027	.57	.56	.056	.003	.015	.007	.018	.005	.009	150	115
Mar. 11	7.35 A.M.–3.54 P.M.	59	1.52	1.06	.059	.011	——	.59	.58	.054	.001	.015	.010	.005	.002	.010	135	110
Mar. 11–12	3.54 P.M.–10.20 A.M.	58	1.43	1.08	.029	.009	.024	.58	.58	.050	.009	.019	.005	.017	.004	.012	140	110
Mar. 12–13	10.20 A.M.–6.59 A.M.	52	1.44	1.01	.053	.009	.024	.56	.60	.045	.022	.001	.005	.018	.003	.010	115	115
Mar. 13–14	6.59 A.M.–7.45 A.M.	52	1.39	0.92	.069	.013	.035	.56	.60	.062	.009	.002	.013	.056	.002	.015	115	125
Mar. 14	7.45 A.M.–2.50 P.M.	64	1.48	0.88	.096	.018	——	.64	.65	.091	.035	.001	.018	.072	——	——	170	——
Mar. 14–15	2.50 P.M.–9.00 A.M.	63	1.47	0.90	.085	.016	.009	.54	.58	.082	.038	.006	.017	.076	tr.	.016	135	155

[1] Steers C and D were fed 9000 grams of hay each, November 11 to December 20, 1923, and thereafter 4500 grams each until March 2, 1924. Body-weight: C, December 20, 693 kg.; D, December 21, 687 kg. C and D, March 2, 1924, 636 and 621 kg., respectively. No food given after 7 A.M., March 3, until 10.30 A.M., March 13 for C, and 10 A.M., March 12 for D. Water allowed as desired. The data between the horizontal cross lines represent those obtained during the period without food.

[2] Not corrected for total creatinine.

TABLE 2

THE CONSTITUENTS OF THE URINE OF STEER D AS AFFECTED BY FASTING[1]
(Amounts per hour)

DATE	TIME	VOLUME	TOTAL NITROGEN	UREA NITROGEN	AMMONIA NITROGEN	AMINO-ACID NITROGEN	HIPPURIC ACID NITROGEN	PREFORMED CREATININE	TOTAL CREATININE	INORGANIC SULPHATE (S)	ETHEREAL SULPHATE (S)	NEUTRAL SULPHUR (S)	PHENOLS		ACETONE AND DIACETIC ACID	β-OXYBUTYRIC ACID	TOTAL FIXED BASES 0.1 N	ORGANIC ACIDS[2] 0.1 N
													FREE	CONJUGATED				
1924		CCMS.	GRAMS	GRAMS	GRAMS	GRAMS	GRAMS	GRAMS	GRAMS	GRAMS	GRAMS	GRAMS	GRAMS	GRAMS	GRAMS	GRAMS	CCMS.	CCMS.
Mar. 2–3	2.00 P.M.–1.45 P.M.	159	1.18	——	——	0.224	0.318	——	——	0	0.095	0.021	0.096	0.113	tr.	0.051	785	485
Mar. 3	1.45 P.M.–3.30 P.M.	123	1.24	0.41	0.091	.043	——	0.57	0.59	0.081	.022	.021	.184	.035	tr.	——	1045	——
Mar. 3–4	3.30 P.M.–6.51 A.M.	180	1.09	.37	.034	.032	.174	.70	.70	.003	.090	0	.043	.114	tr.	.024	550	255
Mar. 4	6.51 A.M.–3.41 P.M.	423	1.34	.92	.047	.019	.138	.66	.70	0	.109	.014	.040	.086	tr.	.022	605	235
Mar. 4–5	3.41 P.M.–6.43 A.M.	67	1.29	.71	.020	.012	.072	.69	.64	0	.067	.003	.024	.090	tr.	.012	155	145
Mar. 5	6.43 A.M.–5.01 P.M.	74	1.52	.91	.037	.014	.066	.70	.67	.005	.058	.001	.045	.056	.017	.015	135	150
Mar. 5–6	5.01 P.M.–6.32 A.M.	46	1.36	.77	.041	.010	.039	.65	.62	0	.050	.001	.019	.055	tr.	.014	65	125
Mar. 6	6.32 A.M.–5.10 P.M.	175	1.72	.98	.132	.013	.045	.56	.61	0	.040	.018	.029	.047	tr.	.008	70	140
Mar. 6–7	5.10 P.M.–6.32 A.M.	93	1.30	.81	.056	.008	.028	.54	.52	.003	.022	.015	.024	.019	tr.	.006	35	110
Mar. 7	6.32 A.M.–2.14 P.M.	203	2.78	1.63	.123	.017	.054	1.03	1.05	0	.040	.041	.054	.018	.005	.014	85	220
Mar. 7–8	2.14 P.M.–6.34 A.M.	95	1.82	1.23	.107	.010	.032	.67	.59	0	.022	.023	.013	.030	.004	.010	45	130
Mar. 8	6.34 A.M.–2.17 P.M.	82	1.86	1.11	.131	.015	.034	.65	.56	0	.019	.015	.025	.013	.008	.014	65	140
Mar. 8–9	2.17 P.M.–6.45 A.M.	62	1.70	1.04	.078	.011	.027	.62	.63	0	.023	.018	.004	.002	.003	.011	65	130
Mar. 9	6.45 A.M.–2.00 P.M.	153	1.95	1.17	.104	.013	.030	.74	.76	0	.013	.044	.014	.024	tr.	.016	110	155
Mar. 9–10	2.00 P.M.–7.05 A.M.	126	1.83	1.20	.105	.012	.026	.66	.66	0	.005	.042	.012	.019	.004	.014	105	135
Mar. 10	7.05 A.M.–1.47 P.M.	110	1.66	1.10	.095	.015	.025	.59	.60	0	.006	.042	.022	.005	tr.	.015	105	120
Mar. 10–11	1.47 P.M.–6.32 A.M.	88	1.55	1.15	.066	.009	.023	.56	.57	0	.005	.029	.024	.013	.003	.010	115	110
Mar. 11	6.32 A.M.–11.20 A.M.	92	1.76	1.25	.107	.023	——	.62	.60	0	.008	.022	.014	.012	.008	.014	130	125
Mar. 11–12	11.20 A.M.–7.50 A.M.	84	1.59	1.14	.079	.012	.024	.68	.66	0	.005	.031	.008	.018	.006	.017	105	120
Mar. 12	7.50 A.M.–2.47 P.M.	257	2.25	1.41	.169	.018	.046	.94	.99	0	.020	.056	.019	.046	.010	.021	190	210
Mar. 12–13	2.47 P.M.–5.54 A.M.	66	1.14	0.62	.060	.009	.032	.53	.57	0	.050	.004	.017	.024	tr.	.012	90	115
Mar. 13	5.54 A.M.–3.15 P.M.	102	1.85	.93	.148	.019	.086	.88	.91	.013	.058	.015	.023	.061	.003	.017	175	205
Mar. 13–14	3.15 P.M.–7.46 A.M.	59	1.02	.48	.091	.018	.083	.58	.61	.053	.032	.001	.029	.046	tr.	.011	150	150
Mar. 14	7.46 A.M.–1.45 P.M.	120	1.38	.48	.100	.020	.135	.72	.76	.136	.050	.014	.049	.072	tr.	.027	385	240
Mar. 14–15	1.45 P.M.–5.58 A.M.	254	1.07	.41	.056	.035	.171	.66	.68	.055	.099	.015	.037	.071	tr.	.026	610	260

[1] See details as to food, fast and body-weight for D in footnote 1, table 1.
[2] Not corrected for total creatinine.

caused considerable variation. As thymol and chloroform failed to keep the composition of the samples constant, the urines were collected in bottles containing hydrochloric acid and then made acid to litmus; this method proved very satisfactory.

With *C* the volume and the constituents of the urine were much more regular than those for *D*, probably because the emptying of the bladder was more nearly complete with the former. The total nitrogen indicates that these animals were on a low nitrogen plane as a result of the preceding undernutrition. This may be seen by comparable calculations which show that with like conditions of fasting and output, the nitrogen elimination with man would be but 5 or 6 grams per day.

The percentage of total nitrogen as ammonia ranged with steer *C* from 1.1 to 6.8 per cent during the period without food, and for steer *D* from 1.5 to 7.7 per cent, the majority of the values for *C* being under and for *D* over 4 per cent, with a slight tendency with *D* to an increase in the latter part of the fast. Thus the ammonia-nitrogen results gave no indication of acidosis.

The urea-nitrogen with both steers was low at first but rose rapidly to a general level which was slightly increased throughout the fast. With *C* the proportion of total nitrogen as urea ranged from 26.4 to 75.4 per cent and after March 4 was over 50 per cent. With *D* the values ranged from 33 to 74.3 per cent. Apparently a smaller proportion of the nitrogen is eliminated as urea when the animal is fed than when he is fasting.

The proportion of total nitrogen as hippuric acid decreased rapidly, ranging with *C* from 27.5 on the first day to 1.7 per cent, being under 2 per cent after March 5–6. With *D*, it was from 26.8 to 1.5 per cent, rising immediately when food was again given. The results indicate that this constituent may be eliminated even when no food is received, and may be considered as derived from endogenous sources.

The amino-acid nitrogen with both animals was 18 to 19 per cent of the total nitrogen at the beginning and fell rapidly. From March 4 to 6 to the end of the fast it was generally under 1 per cent, but rose when food was again taken.

The preformed and total creatinine for both steers had a high degree of uniformity throughout the fast; the few high values cannot be accounted for. The preformed creatinine nitrogen for *C* varied between 12.4 and 25.4 per cent, and for the most part was under 15 per cent of the total nitrogen.

The creatinine coefficient (the 24-hour creatinine divided by the body-weight) was for *C* usually 20 to 23 mgm. per kg. of body-weight and for *D* between 21 and 27 mgm. That for the fasting man studied by Benedict[4] had much the same range (18.1 to 25.3 mgm). Considering the differences in body-weight, and types of metabolism and diet, the like degree of constancy with both steers and man is striking. The difference between

the total and preformed creatinine, which is doubtless creatine, is fairly small and at times absent. In contrast with man and some other animals, the fasting apparently did not increase the creatine elimination.

If we compare the nitrogen constituents found with steers with those obtained by Folin[5] with a man on a nearly protein-free diet (starch paste, and cream), we find that the total nitrogen corresponds to many of his values, that the ammonia-nitrogen results were in a considerable number of cases lower than those of Folin, that the elimination of urea-nitrogen approximates the Folin results, but was generally lower, and that the creatinine values were slightly higher.

Since the phenols are considered to be due to fermentation, it would be expected that during fasting they would practically disappear unless they were derived from body material or putrefaction continued for a long time after food had been withdrawn. While the amounts obtained gradually lowered as the fast progressed, neither the free nor the conjugated phenol entirely disappeared and presumably there was still undigested residue present with fermentation.

The percentages of sulphur in the three forms determined vary widely in both quantity and distribution, as was likewise reported for earlier determinations in this coöperative study.[1] The inorganic sulphate appeared late in the fast, especially with *D*, and increased thereafter to the end of the observations. The ethereal sulphate fell gradually but began to rise with the resumption of food. From the course of these two factors, it would appear that putrefaction continued much longer with *D* than with *C*, but with both continued for some time. In contrast to values obtained in earlier fasts, both the neutral sulphur and the nitrogen-sulphur ratio were variable. With steer *C* the ratio was at first 9.8, rose to 57.4, and fell again to 7.1. The general course with *D* was the same, except that the high values continued over a longer period, the range being from 10.2 for the first day to a maximum of 58.9. The cause for this rise and fall cannot be explained. The percentage distribution of the sulphur varied widely.

Since the fixed bases come from the inorganic elements in the food or body material, the rapid fall to be seen in these values would be expected. The increase in elimination which appeared later in the fast might be interpreted to represent either a diminution in body fluids or a destruction of bone or other organized body material. The organic acids (uncorrected for total creatinine), like the fixed bases, show some fall to a more or less general level after the first few days, though with *D* there was more variation.

Acetone, diacetic acid and beta-oxybutyric acid, constituents especially common with fasting man, were determined for both steers. With steer *C* the acetone bodies were extraordinarily small throughout and the beta-oxybutyric acid decreased rather than increased. With *D*, both the

acetone bodies and the beta-oxybutyric acid eliminations, while slightly higher than with *C*, were not significantly high. These results, taken in connection with the values found for ammonia-nitrogen, indicate that these animals, like the dog and the cat, exhibited no tendency toward ketosis during fasting. Apparently the keto-anti-ketogenic ratio holds for man but not for these animals. Unpublished figures show that the respiratory quotient of the steers fell as rapidly to a fasting value as with man, consequently absence of ketosis was not due to the fact that the proportion of katabolized carbohydrate to fat was high or that the animals were not burning fat. This suggests the possibility that this ratio of carbohydrate to fat is not necessarily the cause of ketosis but that it is due to some other factor, man being one of the animals which exhibit this condition when subjected to high fat metabolism such as occurs with a carbohydrate-free diet or during fasting.

(Details of the analyses of the urines of other fasts are to be published in a forthcoming monograph of the Carnegie Institution of Washington by F. G. Benedict and E. G. Ritzman, to whom I am indebted for the possibility of securing the material with which to make this study. The analyses were made by Miss Dorothy M. Tibbetts, and Messrs. Philip P. Saponaro and Edward S. Mills.)

[1] Benedict and Ritzman, *Carnegie Inst. Wash. Pub.*, No. 324, 1923; Carpenter, *J. Biol. Chem.*, 1923, **55**, p. iii.

[2] The phenols were determined by the method of Tisdall (Tisdall, *J. Biol. Chem.*, 1920, **44**, p. 409); the organic acid by the method of Van Slyke and Palmer, *Ibid.*, 1920, **41**, p. 567; the methods for the other determinations are given in Folin's Manual (Folin, Laboratory Manual of Biological Chemistry, New York, 1922).

[3] Grouven, Physiologisch-chemische Fütterungsversuche. Zweiter Bericht über die Arbeiten der agrikulturchemischen Versuchsstation zu Salzmünde, Berlin, 1864, p. 167, et seq.

[4] Benedict, *Carnegie Inst. Wash. Pub.*, No. 203, 1915, p. 267.

[5] Folin, *Amer. J. Physiol.*, 1905, **13**, p. 66.

THE TORSION PROBLEM OF CURVED BEAMS

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Given a beam in elastic equilibrium whose mean fibre is a continuous curve and which is acted upon by couples of arbitrarily directed axes, the problem which we set is to determine:

I. The system of forces acting on any section normal to the mean fibre.

II. The stresses existing in this section and the limiting of those stresses.

Determination of the forces acting on a section.—We first consider a section ABC of a plane curved beam represented in Fig. 1, by its mean fibre, with a torsional moment applied at A in a plane normal to the mean fibre. The moment applied at A is represented by the vector $\overline{AA}'$ following the ordinary conventions of mechanics. To determine the forces acting on a Section B normal to the mean fibre at B, we prolong $\overline{AA}'$ until it intersects the trace BD of the section. At D we carry $\overline{DD}' = \overline{AA}'$ which we decompose, in conformity with the de St. Venant principle of elastic equivalence of statically equivalent systems, into its components DD'' and DD''', tangential and normal, respectively, to the trace BD. Then the vectors $\overline{BB}'$ and $\overline{BB}''$ brought into B, equal $\overline{DD}'''$ and $\overline{DD}''$ are the vectors representative of the moments acting on the section B.

In the case of a bar of rectangular cross section, figure 2, the vector $\overline{BB}'$ will be representative of the moment imposing torsion, upon the section $PQRS$, about an axis xx tangent at B to the mean fibre; the vector $\overline{BB}''$ is representative of the moment imposing simple flexion upon the same section about the neutral axis yy, $P'Q'R'S'$ being the position taken by the section $PQRS$ under the action of the bending moment.

The above known method can be readily extended to a beam whose mean fibre is any continuous curve which, for instance, projects in $ABCOD$ in figure 3. Let this beam be acted upon by moments represented by their respective vectors which project in AA', BB', etc. If we consider at O a normal section to the mean fibre and l_k' and m_k' be the cosines of the angles made by the k vectors $\overline{AA}'$, $\overline{BA}'$, etc., with the plane of the section O a normal to this plane, the forces acting upon this section O may be represented by the two moment-vectors M_1, M_2, where

$$M_1 = \sum_1^k l_k' M_k, \qquad M_2 = \sum_1^k m_k' M_k. \tag{1}$$

The summation extends over the k moments acting on the isolated section $ABCO$.

The vector M_1 is representative of the moment imposing torsion upon the

section about an axis tangent to the mean fibre. The vector M_2 is representative of a moment of bending around an axis parallel to the vector M_2. If the cross section O admits two principal axes of symmetry y and z and the angular directions of the vectors M_k with the directions of y and z and a normal to the section be, respectively, m_k, n_k, l_k, equations (1) will take the form:

$$M_1 = M_x = \sum_1^k l_k M_k, \qquad M_y = \sum_1^k m_k M_k, \qquad M_z = \sum_1^k n_k M_k \quad (2)^1$$

The vectors M_y and M_z are representative of the moments of bending around the two principal axes of symmetry of the section.

The two following particular cases will serve as illustrative verifications:

1. Consider, figure 4, a beam curved in the plane of the figure into quarter circle and leaning by its plane and BB' against the plane yy. Let M_t be a torsional couple applied at AA' in the plane normal to the mean fibre. We shall obtain for the section BB':

$$M_x = M_z = O, \; M_y = M_t$$

that is: the section BB' of the beam will only undergo a simple flexion around the neutral axis yy. The correctness of the result is readily visualized.

2. Consider, figure 5, a cylindrical bar curved into a quarter circle from AA' to BB' and straight from BB' to CC', the straight part being engaged in a straight cylindrical sleeve. We apply at AA' a flexional moment represented by the vector moment M_f.

We shall obtain for BB'

$$M_y = M_z = 0, \qquad M_x = M_f$$

that is: the portion from BB' to CC' undergoes a torsional action which will result in a smooth counterclock-wise rotation of the bar around the axis xx, parallel to the mean fibre.

Calculation and Limiting of the Stresses.—The stress components due to each of the moments M_x, M_y and M_z can now be calculated separately and then superposed. We shall see that *we thereby arrive at a solution of the problem of the stresses in plane and skew beams acted upon by couples for all cases where separate solutions exist for the problem of torsion of the straight beams and of the flexion of the plane curved beams.* We postulate, however, the validity of the principle of superposition of forces and their effects.

We shall treat here for demonstration the case of a curved bar of rectangular cross-section.[2]

Let M_x, M_y and M_z be the torsional and flexional moments in the section considered. The stresses due to the moments M_x, M_y and M_z are obtained, respectively, as follows:

a. According to the method of von Bach, which constitutes the approxi-

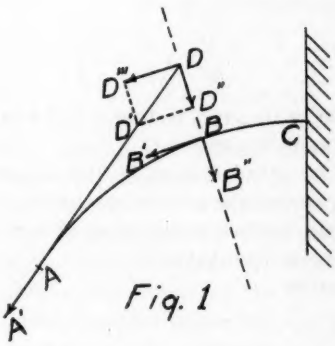

Fig. 1

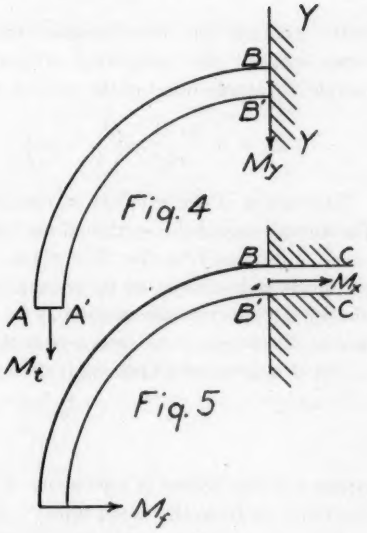

Fig. 4

Fig. 5

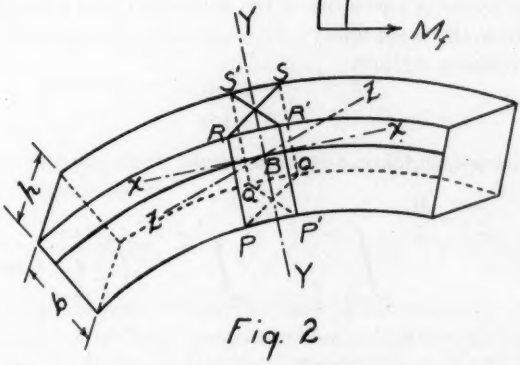

Fig. 2

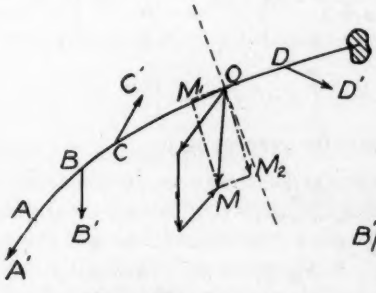

Fig. 3

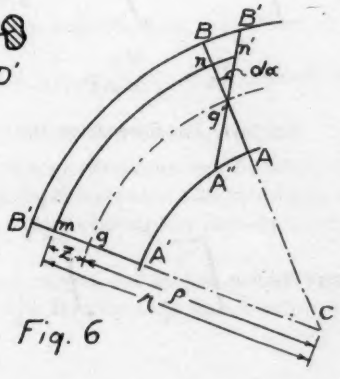

Fig. 6

mate solution for the torsional stresses in straight bars of rectangular cross section, the tangential stresses[3] X_y and X_z due to the torsional couple M_x at any point of the section of coördinates (x, y) are:

$$X_y = +\frac{36\,M_x}{b^3h^3}\,z\left(\frac{b^2}{4} - y^2\right), \qquad X_z = -\frac{36\,M_x}{b^3h^3}\,y\left(\frac{b^2}{4} - z^2\right)$$

The origin of the axes are at the center of the cross section, b and h are the dimensions of the section along the z and the y axes.

b. Let figure 6 be the XOZ plane. Due to the moment M_y, the section AB undergoes a rotation by an angle $d\alpha$ around the axis yy—projecting in g'—with respect to the section $A'B'$. The normals to the mean fibre in g and g', where $gg' = ds$, intersect at the center of curvature C.

The dilatation of a fibre mn is then given by:

$$\frac{nn'}{mn} = \frac{z\,d\alpha/ds}{1 + z/\rho}$$

where ρ is the radius of curvature of the mean fibre and z the distance of the fibre mn from the mean fibre.

The corresponding stress is:

$$X'_x = E\frac{z\,d\alpha/ds}{1 + z/\rho}$$

where E is Young's modulus of elasticity and

$$\frac{d\alpha}{ds} = \frac{M_y}{E}\,\frac{1}{\displaystyle\int_{z=-\frac{h}{2}}^{z=+\frac{h}{2}}\ \int_{y=-\frac{b}{2}}^{y=+\frac{b}{2}} \frac{z^2\,dy\,dz}{1 + \frac{z}{\rho}}}$$

As:[4]

$$\int_{z=-\frac{h}{2}}^{z=+\frac{h}{2}}\ \int_{y=-\frac{b}{2}}^{y=+\frac{b}{2}} \frac{z^2\,dy\,dz}{\rho + z} = b\rho\left[\rho\log\frac{\rho + \frac{h}{2}}{\rho - \frac{h}{2}} - h\right] = K_1$$

We obtain: $X'_x = \dfrac{M_y z}{(\rho + z)K_1}$

c. Similarly, the fibre stress due to the moment M_z is:

$$X''_x = \frac{M_z y}{(\rho + z)K_2}$$

where $K_2 = \displaystyle\int_{z=-\frac{h}{2}}^{z=+\frac{h}{2}}\ \int_{y=-\frac{b}{2}}^{y=+\frac{b}{2}} \frac{y^2\,dz\,dy}{\rho + z} = \frac{b^3}{12}\log\frac{\rho + \frac{h}{2}}{\rho - \frac{h}{2}}$

We thus find that at any point of the section there exist the following components of stress:

$$X_y = -\frac{36\,M_x}{b^3h^3}\,z\left(\frac{b^2}{2} - y\right), \qquad X_z = +\frac{36\,M_y}{b^3h^3}\,y\left(\frac{h^2}{4} - z\right)$$

$$X_x = X'_x + X''_x = \frac{M_y z}{(\rho + z)K_1} + \frac{M_z y}{(\rho + z)K_2}$$

This permits the separate limitation of the normal stresses X_x and the tangential stresses X_y and X_z.

If one desires to limit either the magnitude of the principal stresses, and thereby the magnitude of the maximum normal stress—method of Lamé—or the greatest difference of the principal stresses—method of Coulomb and Tresca—the values of the principal stresses result from the following:

The equation connecting the principal stresses with the components of the stresses here reduces to:

$$\varphi^3 - X_x\varphi^2 - \varphi\,(X_z^2 + X_y^2) = 0$$

and the roots give the following values for the principal stresses:

$$\varphi_1 = 0, \qquad \varphi_2 = \tfrac{1}{2}X_x + \tfrac{1}{2}\sqrt{X_x^2 + 4(X_z^2 + X_y^2)},$$

$$\varphi_3 = \tfrac{1}{2}X_x - \tfrac{1}{2}\sqrt{X_x^2 + 4(X_z^2 + X_y^2)}$$

Moreover, the directions of the three planes of principal stress are given by the direction cosines:[5]

$$l = \frac{\varphi}{\sqrt{\varphi^2 + X_y^2 + X_z^2}}, \qquad m = \frac{X_y}{\sqrt{\varphi^2 + X_y^2 + X_z^2}}, \qquad n = \frac{X_z}{\sqrt{\varphi^2 + X_y^2 + X_z^2}},$$

where φ is successively replaced by the values found for φ_1, φ_2, φ_3.

If one desires to limit the deformation e—method of Poncelet and de St. Venant—or the corresponding fictitious stress Ee—E being Young's modulus of elasticity—the maximum deformation will be obtained as follows:

The expression of e in a direction l, m, n, here becomes:

$$e = \frac{1}{E}\left\{\left[l^2(1+\sigma) - \sigma\right]X_x + 2l(1+\sigma)(nX_z + mX_y)\right\}$$

where σ is Poisson's ratio.

The values of l, m, n for which e becomes a maximum, are determined by equating the partial derivatives of the expression for e with respect to l, m and n to zero and the maximum deformation obtained by introducing those values in the original expression for e.

Similar treatment as above has been carried out by the authors for circular and elliptical cross sections. It is thereby seen that a solution for

the torsional problem of curved beams is reached also for those sections, with no further limitations than those existing in the solutions of the corresponding torsional problem in straight members, and the postulation of the principal of superposition.

If the forces applied are not pure couples, they can in each section be reduced to a couple and a force. This force will in general cause in the further sections an additional bending around the y and z axes, which will be added to the expressions of M_y and M_z in equations (2).

This method can also be used for non-prismatic beams provided the variations of profile are sufficiently slow as to make the disturbance caused thereby on the stress distribution negligible.

[1] The moment vectors M_x, M_y, M_z could also be obtained by graphical addition of the vectors $\overline{AA'}$, $\overline{BB'}$, etc., and subsequent decomposition of the resultant vector into the x, y and z components.

[2] For complete treatment of curved members with circular, elliptical and rectangular cross-sections, see Paul Heymans and W. J. Heymans, *J. Math. and Physic, M. I. T.*, **4**, 1, 1924.

[3] The notations used are those of A. E. H. Love, "Treatise on the Mathematical Theory of Elasticity."

[4] Heymans, *loc. cit.*

[5] *Loc. cit.*

A NEW INTERPRETATION OF THE HEREDITARY BEHAVIOR OF SELF-STERILE PLANTS

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Among the results appearing from the studies of self-sterility in the genus Nicotiana carried on in this laboratory are the following:

1. Self-sterility is a Mendelian recessive in crosses between self-fertile and self-sterile species. Self-sterile plants, therefore, breed true to self-sterility.

2. Self-sterility in such self-sterile species as *Nicotiana alata* Lk. and Otto and *Nicotiana forgetiana* Hort. (Sand.), or their hybrids, is controlled by numerous inherited factors. Like combinations of these factors give like groups of individuals as far as their compatibility with each other is concerned, just as like factorial combinations which lie at the basis of morphological differences give groups which are alike in those characters.

3. All members of a given self-sterility group are cross-sterile with each other in reciprocal crosses, and all are cross-fertile with members of every other group, with certain limitations as shown in the new data presented in this paper.

4. Not less than fifteen and possibly as high as twenty-five such intra-sterile, inter-fertile groups are found in the material used in these studies.

5. The basis upon which these variations in compatibility rest are hereditarily controlled reactions whereby some pollen tubes (compatible matings) show a rapidly accelerated rate of growth, while other pollen tubes (incompatible matings) show a slow and steady rate of growth which ordinarily does not permit fertilization to occur within the life of the flower.

6. Conditions can be so controlled that (*a*) the length of "life" of the flower is prolonged, (*b*) the rapidity of the growth of the pollen tube is stimulated, and (*c*) the opportunity for pollen-tube action is lengthened.

By taking advantage of various methods mentioned under heading No. 6, chief of which is simply the pollination of the young bud, selfed strains of our self-sterile hybrids between *N. alata and N. forgetiana* have been carried through twelve generations. This lengthy manipulation was carried out with a purpose. If the factors controlling the behavior of these self-sterile plants are distributed according to the regular Mendelian procedure,[1] successive selfings should tend to eliminate heterozygotes. If, furthermore, compatibility between self-sterile plants is brought about by the existence of constitutional differences in certain factors, then one ought to be able to produce near homozygous populations by selfing which would consist of very few intra-sterile, inter-fertile classes.

This expectation has been realized. Numerous families have been raised which consist of two intra-sterile, inter-fertile groups. A smaller number of families have been found to consist of a single group of this type, each individual being cross-sterile with every other individual, though fertile with individuals in other families.

Some twenty-five thousand pollinations have now been made in testing the reactions of this type of family. Data on a few of these families have been published in the sixth paper of these "Studies on Self-sterility"[2] by Dr. Edgar Anderson. We shall call attention first to a peculiarity in these results as follows:

NAME	ORIGIN	CLASSES CROSSED ♀ × ♂	CLASSES RESULTING AND NUMBER OF PLANTS OF EACH	
Family A O	AA-2 × AA-13	a × b	6-b	4-c
Family A L	O-32 × O-46	a × b	8-b	12-c
Family A T	P-27 × O-2	a × b	11-b	6-c
Family A H	L-18 × L-20	a × c	8-b	9-c
Family A I	M-31 × M-25	a × c	9-b	11-c
Family A R	R-38 × L-24	a × c	13-b	7-c
Family A P	R-20 × R-13	c × a	7-a	9-b
Family B P	N-7 × R-12	b × c	6-a	4-c
Family B O	R-2 × O-2	c × b	4-a	6-b

In all of these cases it will be noticed that the class to which the female parent belongs is absent in the progeny. We are dealing with three classes,

a, b and c. Each individual of any one of these classes is sterile reciprocally with any other individual of the class to which it belongs and is fertile reciprocally with every individual belonging to either of the other two classes. But the progeny resulting from reciprocal crosses is different. Class "a" × Class "c" gives "b" individuals and "c" individuals only, while Class "c" × Class "a" gives "a" individuals and "b" individuals only. Two classes always appear in what seems to be an approximation to equal numbers, *but the class of the mother is never represented.*

These results have been corroborated numerous times by data obtained since Dr. Anderson's figures were reported, as the following table shows. The classes are here called X, Y and Z, though they prove to be the same classes with which Anderson dealt, when tested by the proper crosses.

NAME	ORIGIN	CLASSES CROSSED ♀ × ♂	CLASSES RESULTING AND NUMBER OF PLANTS OF EACH	
			Class Y	Class Z
Family Q_1 N_1	Q_1 x N_1	Class X × Class Y	21	19
Family R_3 E_2	R_3 x E_2	Class X × Class Y	13	15
Family R_3 R_1	R_3 x R_1	Class X × Class Y	21	13
		Total	55	47
			Class X	Class Z
Family E_2 R_3	E_2 x R_3	Class Y × Class X	21	9
Family O_1 O_2	O_1 x O_2	Class Y × Class X	23	17
Family Q_2 Q_1	Q_2 x Q_1	Class Y × Class X	18	20
		Total	62	46
			Class Y	Class Z
Family O_2 L_1	O_2 x L_1	Class X × Class Z	8	16
Family Q_1 L_1	Q_1 x L_1	Class X × Class Z	21	18
Family R_3 C_{13}	R_3 x C_{13}	Class X × Class Z	20	16
		Total	49	50
			Class X	Class Y
Family L_1 O_2	L_1 x O_2	Class Z × Class X	22	16
Family L_1 Q_1	L_1 x Q_1	Class Z × Class X	8	12
		Total	30	28
			Class X	Class Z
Family N_1 L_1	N_1 x L_1	Class Y × Class Z	11	8
Family Q_2 L_1	Q_2 x L_1	Class Y × Class Z	29	9
Family R_1 C_{13}	R_1 x C_{13}	Class Y × Class Z	12	20
Family R_1 L_1	R_1 x L_1	Class Y × Class Z	16	24
		Total	68	61
			Class X	Class Y
Family R	C_{13} x E_2	Class Z × Class Y	10	10
Family L_1 O_1	L_1 x O_1	Class Z × Class Y	23	13
Family L_1 Q_2	L_1 x Q_2	Class Z × Class Y	23	17
		Total	56	40

It is clear that the same phenomenon is exhibited. When individuals of any two classes are crossed, *two* classes appear in approximately equal

numbers, but the class to which the female parent belongs is never represented.

X♀ × Y♂	gives	Y and Z	Z♀ × X♂	gives	X and Y
Y♀ × X♂	gives	X and Z	Y♀ × Z♂	gives	X and Z
X♀ × Z♂	gives	Y and Z	Z♀ × Y♂	gives	X and Y

At first sight one might suppose that these results, differing as they do from any hitherto obtained in genetic work, are due to a novel type of gamete distribution; yet such a supposition is rendered unlikely by two facts. Morphological characters in these varieties appear to be transmitted in the normal Mendelian manner; furthermore the ratios in which the classes appear here are strikingly like those expected when F_1 monohybrids are backcrossed with pure recessives.

To be brief, we believe that the inheritance in these cases is of the ordinary type, but that there are physiological limitations to the opportunity for fertilization.

Assume that in these populations there are three allelomorphs, S_1, S_2 and S_3; and that

Class X = $S_1 S_3$, Class Y = $S_1 S_2$, Class Z = $S_2 S_3$.

Assume further that a plant affords stimulus only to pollen which bears sterility factors other than its own. Thus a Class Z plant ($S_2 S_3$) affords proper stimulus only to gametes carrying factors other than S_2 or S_3. In other words, of the three factors present in these various populations, the Z plants afford stimulus only to S_1 gametes.

Now what happens when one crosses a Z female by an X male? Only the S_1 pollen is stimulated and functions. The Z female being $S_2 S_3$ in composition stimulates neither S_2 nor S_3 pollen grains. The resulting progeny consists therefore of two classes, $S_1 S_2$ (Class Y) and $S_1 S_3$ (Class X) in equal numbers. In the reciprocal cross, X female ($S_1 S_3$) by Z male ($S_2 S_3$), the X female again affords stimulus only to gametes-bearing factors other than its own—this time, the S_2 pollen. The resulting progeny is therefore solely of the two classes Y ($S_1 S_2$) and Z ($S_2 S_3$).

The results to be expected according to this hypothesis are in accordance with the data in hand in that (*a*) the class of the female parent is always absent in the progeny, (*b*) the progeny of reciprocal crosses is not the same two classes, and (*c*) the progeny consists of two classes other than the female parent in approximately equal numbers regardless of which of the other two classes serves as the male parent.

This hypothesis can be tested further by selfing individuals or by crossing individuals belonging to the same class. This is done, as explained above, by taking advantage of the "pseudo-fertility" which occurs when non-compatible pollen is given the *opportunity* to effect fertilization. In practice this is usually done by pollinating young buds.

Family R_1 (Class Y = S_1 S_2) selfed produced 32 plants. Of these individuals, 19 were Class Y (S_1 S_2), and showed typical Class Y behavior. Two new classes also appeared. Of the first class, Class Z Y, there were 7 individuals. These plants were fertile as females with all classes except their own, and sterile as males with their own class and also with classes Z and Y. The second class, Class XY of which there were 6 individuals, was fertile as a female with all classes except its own, and was sterile as a male with its own class and also with Classes X and Y.

Similarly Family Q_2 (Class Y = S_1 S_2) selfed gave 24 individuals of Class Y, 14 individuals of Class Z Y and 1 individual of Class X Y.

According to the selective pollen stimulation hypothesis, what happens is this:

Class Y (S_1 S_2), when selfed in the very young bud, gives

1 S_1 S_1 (Class XY) + 2 S_1 S_2 (Class Y) + 1 S_2 S_2 (Class Z Y).

The S_1 S_1 plants (Class X Y) are fertile as females with

1. Class Z (S_2 S_3) because they accept both S_2 and S_3 pollen,
2. Class X (S_1 S_3) because they accept S_3 pollen, and
3. Class Y (S_1 S_2) because they accept S_2 pollen.

They are fertile as males with Class Z (S_2 S_3) because Z accepts S_1 pollen; but, they are sterile as males with both Class X (S_1 S_3) and Class Y (S_1 S_2), because neither of these classes accept S_1 pollen.

For analogous reasons, individuals of Class Z Y (S_2 S_2) are fertile as females with Classes Z, X and Y, but are sterile as males with Classes Z and Y.

In two other families Y plants were crossed with each other. This should give the same results as when Y's are selfed. The first cross (No. E_2 × No. R_1) gave 17 Class Y + 5 Class Z Y + 1 Class X Y. The second cross (No. R_1 × No. E_2), a reciprocal, gave 21 Class Y + 12 Class Z Y + 6 Class X Y. The total results from Class Y × Class Y(S_1 S_2 × S_1 S_2), therefore are 81 Class Y (S_1 S_2) + 38 Class Z Y (S_1 S_1) + 14 X Y (S_2 S_2). There is a marked deficiency in Class X Y which we are not able yet to ascribe definitely either to differential pollen stimulation or to a lethal effect when S_2 fertilizes S_2. It seems more likely in this particular case to be due to the first cause cited.

There is one bit of evidence on this point. According to the hypothesis, when a Y female (S_1 S_2) is pollinated by an X male (S_1 S_3) only the S_3 pollen should function, resulting in progeny of two classes, viz., Class X (S_1 S_3) and Class Z (S_2 S_3). Such results are normally obtained. But if pollinations are made in the *very young* bud, some of the S_1 pollen might be expected to have the opportunity to function. One family has been obtained from such a pollination, Family R_1 R_3, coming from a pollination of R_1 female of Class Y by R_3 male of Class X. The population obtained consisted of 34 individuals belonging to the following classes, 13 Class X

(S_1 S_3) + 13 Class Z (S_2 S_3) + 4 Class Y (S_1 S_2) + 4 Class X Y (S_1 S_1). According to hypothesis, the cross of S_1 S_2 by S_1 S_3 gave large numbers of Class X (S_1 S_3) and of Class Z (S_2 S_3) because the Class Y female (S_1 S_2) afforded a decided stimulus to S_3 pollen, and gave small numbers of Class Y (S_2 S_1) and Class X Y (S_1 S_1) because very little stimulus was afforded to S_1 pollen. But fertilization with S_1 pollen *was* possible, when the opportunity was forthcoming.

On the other hand, Class Z (S_2 S_3) selfed, which might be expected to give 1 Class Z Y (S_2 S_2) + 2 Class Z (S_2 S_3) + 1 Class Z X (S_3 S_3), excepting as the ratio were disturbed by selective pollen stimulation, did in fact give no plants of Class Z X (S_3 S_3). There were obtained 17 individuals of Class Z (S_2 S_3) and 12 individuals of Class Z Y (S_2 S_2). Likewise, Class X (S_1 S_3), selfed, in two different cases gave 57 individuals of Class X (S_1 S_3) and 20 individuals of Class X Y (S_1 S_1), but not a single individual of Class Z X (S_3 S_3). Since the number of plants in Classes Z Y (S_2 S_2) and X Y (S_1 S_1), respectively, in these two cases showed no particular excess when compared with the heterozygous classes, one must conclude that the heterozygous classes were not composed wholly of S_3 ovules fertilized by S_2 pollen and of S_3 ovules fertilized by S_1 pollen. In each example S_3 pollen must have reached the ovaries and have functioned in the production of heterozygotes. For this reason, it seems as if one were forced to conclude that S_3 is lethal when in the homozygous condition.

[1] Conceivably methods of inheritance which are non-Mendelian might bring about similar results.

[2] The genetic basis of cross-sterility in *Nicotiana*. *Genetics*, **9**, 13–40, 1924.